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HYPERCLASSICAL LOGIC (A.K.A. IF LOGIC) AND ITS IMPLICATIONS FOR LOGICAL THEORY

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Let us assume that you are entrusted by UNESCO with an important task. You are asked to devise a universal logical language, a *Begriffsschrift* in Frege's sense, which is to serve the purposes of science, business and everyday life. What requirements should such a "conceptual notation" satisfy? There are undoubtedly many relevant desiderata, but here I am focusing on one unmistakable one. In order to be a viable *lingua universalis*, your language must in any case be capable of representing any possible configuration of dependence and independence between different variables. For if such a configuration is possible in principle, there is no guarantee that it might not one day show up among the natural, human or social phenomena we have to study.

But how are dependencies and independencies between variables expressed in our familiar logical notation? Every logician worth his or her truth-table knows the answer. Dependencies between two variables are expressed by dependencies between the quantifiers to which they are bound. For instance, in

$$(1) \qquad (\forall x)(\exists y)\ \mathrm{F}[x, y]$$

the variable y depends on x, while in

$$(2) \qquad (\forall x)(\exists z)(\forall y)(\exists u)\ \mathrm{F}[x, y, z, u]$$

z depends on x but not on y, while u depends on both x and y.

But how is the dependence of a quantifier on another one expressed in familiar logical languages? Obviously by occurring in its scope, indicated by the pair of parentheses following it (cf. here Hintikka [1997]). But the nesting of scopes is a transitive and antisymmetrical relation which allows branching only in one direction. Hence other kinds of structures of dependence and independence between variables are not representable in the received logical notation. Such previously inexpressible structures form the subject matter of what has been referred to as independence-friendly (IF) logic. (See Hintikka [1996]; Hintikka and Sandu [1996].) Nor is there anything unnatural or even

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fancy about some such conventionally inexpressible quantifier structures. The oldest and best known type of such previously unspeakable patterns is constituted by what are known as branching quantifiers. The simplest branching quantifier structure is known as the Henkin quantifier. It can be expressed in a self-explanatory nonstandard two-dimensional notation as follows:

$$\begin{array}{l} (\forall x)(\exists y) \searrow \\ \qquad\qquad\qquad\quad \mathrm{F}[x, y, z, u] \\ (\forall z)(\exists u) \nearrow \end{array} \tag{3}$$

The meaning of (3) is obvious. It can be specified by describing a game between two teams which are the "players" of the game in the sense of game theory, the "verifier" and the "falsifier", each consisting of two agents. Each of the two members of the falsification team chooses an individual, say x and z. Then one member of the verification team chooses y on the basis of x alone, and the other one chooses u, knowing only z. The verification team wins if and only if the individuals so chosen satisfy $\mathrm{F}[x, y, z, u]$. The sentence (3) is true if and only if there exists a winning strategy for the verification team.

The study of branching quantifiers was begun by Henkin [1961]. The classical papers include Walkoe Jr. [1970], Enderton [1970], and Barwise [1979]. Independence-friendly logic is often thought of as little more than the study of branching quantifiers. In fact there are other even more interesting quantifier patterns. The simplest is constituted by a pair of mutually dependent quantifiers.

How can the logic of such quantifier patterns be studied? How can they be expressed in a logical language? Interestingly the semantics of independent quantifiers is easier to master than their syntax. The explanation given of the meaning of (3) illustrates the approach known as game-theoretical semantics. It offers a simple and powerful method of studying IF logic. I am assuming familiarity with its main features. (Cf. here e.g., Hintikka and Sandu [1996]). A game-theoretical treatment can be given already to the received first-order logic. This semantics can be extended to IF logic simply by allowing informational independence in the sense of the general theory of games. No other changes are needed.

Does this make IF first-order logic somehow a part of the theory of games? No, what is involved here is the insight that the substantial dependence and independence of variables on each other can be conveniently modeled by informational dependence and independence of quantifiers in the sense of game theory. (Cf. here Hintikka [forthcoming (c)].)

However, the central role of informational dependence in modelling substantial dependence relations motivates a change in the most common way of dealing with informational independence game-theoretically. In the usual treatments of game theory, the informational dimension is introduced by associating with each node of the game tree what is called its information set. This set is the class of all nodes between which the player making the relevant move cannot distinguish on the basis of his or her information. In semantical games and in many other games this informational indistinguishability results from the player's ignorance of certain earlier moves. However, in some games (e.g., bridge considered as a two-person game) it can result in some other way. This extends the applicability of game theory, but in the special case of semantical games it complicates the theory unnecessarily. For this reason, in these games it is reasonable to restrict our attention to the informational dependence of different moves on others, in other words, to consider instead of information sets in the usual sense the sets of other moves on which a given one depends or does not depend informationally. Accordingly, in this paper information sets will not be considered, only dependencies between different moves.

At the same time, game-theoretical semantics can do more than it has been so far drafted to do. What it can do is to provide a method of expressing and of studying the dependence (and the independence) of any quantifier on any other one in a first-order sentence. This requires no sweat. It is part and parcel of the very game-theoretical approach to logical semantics. This can be done by expressing the dependencies in question by means of explicit function variables.

Dependencies between quantifiers in a given sentence are in fact spelled out by what are known as its Skolem functions. A function f that satisfies

(4) $$(\forall x)\ \mathrm{F}[x, f(x)]$$

is a Skolem function of (1). Likewise, the Skolem functions of (2) and of (3) satisfy the following respective requirements ie (dependencies):

(5) $$(\forall x)(\forall y)\ \mathrm{F}[x, y, f(x), g(x, y)]$$

(6) $$(\forall x)(\forall z)\ \mathrm{F}[x, f(x), z, g(z)]$$

The generalization to all formulas is obvious. One can say that while a quantifier sentence like (1) or (2) asserts that certain dependencies hold, its different Skolem functions represent different possible modes of dependence. The pattern of dependence relations is shown by the choice of the argument sets of the different Skolem functions.

Mutually dependent variables likewise have associated with them Skolem functions, for instance, the functions f, g in

(7) $$(\forall x)(\forall y)\ ((x = f(y))\ \&\ (y = g(x))\ \&\ \mathrm{F}[x, y])$$

In game-theoretical terms, arrays of Skolem functions of a sentence S codify winning strategies available for the verifier in a semantical game played with S. Truth is defined as the existence of such a winning strategy. (By the same token, falsity means existence of a winning strategy for the falsifier.) Hence the game-theoretical truth-condition of a Henkin quantifier sentence (3) is

$$(\exists f)(\exists g)(\forall x)(\forall z)\ \mathrm{F}[x, f(x), z, g(z)] \tag{8}$$

and of the simplest sentence with mutually dependent quantifiers (cf. (7))

$$(\exists f)(\exists g)(\forall x)(\forall y)\ ((x = f(y))\ \&\ (y = g(x))\ \&\ \mathrm{F}[x, y]) \tag{9}$$

In general, a first-order sentence (IF or not) is logically equivalent with the second-order sentence that asserts the existence of a full set of its Skolem functions. For instance, (8) is a truth-condition of (3) and (9) is the truth-condition of a first-order sentence with mutually dependent quantifiers. However, at this point the question remains open as to how they can be expressed on the first-order level. I will return to this question in its general form below. Meanwhile, as an interim measure we can introduce a preliminary slash notation. What it does is to exempt a quantifier $(Q_2 y)$ from its dependence on another quantifier $(Q_1 x)$ within whose syntactical scope it occurs by writing it $(Q_2 y/Q_1 x)$. Using this notation, the first-order sentences equivalent with (8) and (9) can be written as follows:

$$(\forall x)(\forall z)(\exists y/\forall z)(\exists u/\forall x)\ \mathrm{F}[x, y, z, u] \tag{10}$$

$$(\forall x)(\forall y)(\exists z/\forall x)(\exists u/\forall y)\ ((x = z)\ \&\ (y = u)\ \&\ \mathrm{F}[x, y]) \tag{11}$$

Game-theoretical semantics might at first look like a recondite way of approaching IF first-order logic. In reality, it is little more than an explication of our own pretheoretical ideas about quantifiers and truth. For if you contemplate what Skolem functions are — and even if you do not contemplate it — obviously their job description is to produce the very "witness individuals" (usually depending on earlier choices of such witness individuals) that show the truth of the quantifier sentence in question. This is in fact illustrated by (4)–(7). One can therefore look upon game-theoretical semantics as being little more than an elaboration of our pretheoretical idea of truth as applied to quantified sentences.

This basic idea, as well as its implementation by means of the notion of Skolem function, can (and must) be extended to propositional connectives in an obvious fashion. For instance, the sentence

$$(\forall x)(\mathrm{A}[x] \vee \mathrm{B}[x]) \tag{12}$$

is equivalent with

$$(\exists f)(\forall x)((\mathrm{A}[x]\ \&\ (f(x) = 0)) \vee (\mathrm{B}[x]\ \&\ (f(x) \neq 0))) \tag{13}$$

Here 0 can be thought of simply as some designated member of the domain. Now (13) is in turn equivalent with

$$(14)\quad (\forall x)(\forall z)(\exists y/\forall z)$$
$$(\exists u/\forall x)(((x = z) \supset (y = u)) \,\&\, (\mathrm{A}[x] \,\&\, y = 0) \vee (\mathrm{B}[x] \,\&\, y \neq 0))$$

The equivalence between (13) and (14) can be seen by noting that (14) is equivalent with

$$(15)\quad (\exists f)(\exists g)(\forall x)(\forall z)(((x = z) \supset (f(x) = g(z))) \,\&$$
$$((\mathrm{A}[x] \,\&\, f(x) = 0) \vee (\mathrm{B}[x] \,\&\, f(x) \neq 0)))$$

Indeed by putting $x = z$ it is seen that in (15) $f = g$. Hence (15) reduces to (14), which is the second-order Skolem equivalent of (13).

In this paper, in the interests of simplicity, I will nevertheless assume that no slashed connectives are present.

A number of general theoretical conclusions can be drawn already at this stage of the game.

(1) The received logic of quantifiers that goes back to Frege and to Russell and Whitehead does not fulfill its job description. It is not an unrestricted general theory of quantifiers. It applies only when the pattern of dependencies and independencies between quantifiers in a sentence satisfies certain conditions. In contrast, IF first-order logic is the general unrestricted logic of first-order quantifiers. For this reason, it should not have been designated by any particular epithet like "independence-friendly". It is simply the natural unrestricted first-order logic. Because of its presuppositions, it is the so-called "ordinary" first-order logic that needs a special moniker. Perhaps we should call it dependence-handicapped or more correctly independence-challenged first-order logic.

In any case, as mutually dependent quantifiers shows, IF logic is not only independence-friendly but also dependence-friendly, in that in it quantifier patterns can be expressed that involve more dependencies than can be expressed in ordinary ("dependence-handicapped") logic.

(2) Frege's and Russell's oversight is an instance of a more general phenomenon. This phenomenon is a defective grasp of the meaning (i.e. semantics) of (standard) quantifiers. An existential or universal quantifier does not receive its entire meaning from its variable's ranging over a class of values. There is another important component to the semantics of quantifiers. This second component is constituted by the relations of dependence and independence between quantifiers. Because of this ingredient in the meaning of quantifiers, their semantics cannot be fully captured by referring only to the class of values they range over. Among the consequences of this insight is the fact that quantifiers cannot be construed as higher-order predicates, Frege

notwithstanding. Such an attempted treatment misses the second (dependence) component in the meaning of quantifiers. For instance, the so-called theory of generalized quantifiers, which is in effect predicated on Frege's unfortunate plan of trying to construe quantifiers as higher-order predicates, is not even a fully satisfactory treatment of ungeneralized quantifiers.

These two components in the meaning of quantifiers are largely independent of each other. One can for instance have a two-sorted quantifier language with one subset of quantifiers ranging over one class of values and another subset over the other. In spite of this, quantifiers from the two subclasses can be dependent and independent of each other without any further restrictions. If you internalize this idea and apply it to two spatially separated classes of values of quantified variables, the idea of "action at distance" can be seen to be virtually a built-in feature or at least a null hypothesis of our usual logical and mathematical languages.

(3) IF logic is not a "non-classical" logic. It is obtained by leaving all the semantical rules (formulated within a game-theoretical framework) unchanged and by merely allowing them to be used in a more flexible manner than before. I would now prefer to call it hyperclassical logic.

Under any term it is the general unrestricted first-order logic. Some philosophers have been so blindly committed to the "ranging over" idea as the whole truth about quantifiers that where this idea fails, as it fails in IF logic, they have jumped to the conclusion that such a logic must somehow be higher-order. This is nonsense by their own criteria, for the only reasonable way of making the first-order vs. higher-order distinction is in terms of the entities one's quantified variables range over. And by this criterion IF first-order logic is indeed first-order.

(4) On the surface IF logic admittedly displays certain features that are not usually considered "classical". The main one is the failure of *tertium non datur.* The negation that is defined by the usual, perfectly "classical" game-theoretical rules is a strong (dual) one which does not obey the law of excluded middle. It is not clear, however, what should be concluded here. The fact that the dual negation results from perfectly traditional semantical rules suggests that the law of excluded middle perhaps is not an integral ingredient of the rightly understood classical conception of logic after all. There are indeed strong indications of that this is indeed the case; see e.g., Hintikka [forthcoming (b)].

The failure of *tertium non datur* must nevertheless be kept in mind. It forces us to recognize sentences which are neither true nor false. It follows that we have to distinguish equivalences that hold only for the truth of their two sides from equivalences that preserve both truth and falsity. In the rest of this paper, the former kind of equivalence is meant unless otherwise indicated. In this sense two sentences are equivalent if and only if they are true in all the same models, but not necessarily true *and false* in the same models. In general, a host of interesting perspectives on the concept of negation are opened by IF logic. They will be examined in a separate paper (see Hintikka [forthcoming (b)]).

But how can we generalize the received logical notation so as to allow arbitrary patterns of dependence and independence? What are for instance the first-order counterparts to (6)–(7)? Earlier, we introduced the interim slash notation. But can it express all possible patterns of dependence and independence between quantifiers? Are there other ways of doing so? In principle, there are different possibilities here. Conceptually the simplest solution would be merely to liberalize the use of brackets (parentheses), allowing the syntactical scope of a quantifier to be a discontinuous part of the sentence in question and allowing the scopes of the quantifiers to overlap only partially. Then one has to index parentheses to show whose scope they are defining. Then (6)–(7) could be expressed as follows:

(16) $$(\forall x)(\forall z)\ {}_{\forall x}((\exists y))_{\forall x}\ {}_{\forall z}((\exists u)_{\forall x}(\mathrm{F}[x,y,z,u])_{\forall x})_{\forall z}$$

(17) $${}_{\forall y}((\forall x))_{\forall y}\ {}_{\forall x}((\forall y)\ {}_{\forall y}(\mathrm{F}[x,y])_{\forall y})_{\forall x}$$

Obviously every possible pattern of dependence and independence between quantifiers can be represented in this way. As examples like (16)–(17) show, this way of dependence indication soon results in formulas that are impossible to read easily. Hence this notation is impractical.

On the theoretical level, the possibility of such a treatment is nevertheless highly instructive. It shows that IF logic is indeed a "classical" one. Nothing new has been added over and above the received first-order logic. The only thing that has happened notationally is that the punctuation of our formulas has been liberalized. Such a loosening up on punctuation does not introduce anything new ontologically or ideologically. Hence the epithet "hyperclassical" can be seen to be fully merited.

By the same token, it is seen that IF first-order logic is indeed first-order. In the step from the ordinary paradigm case first-order logic to IF logic nothing is changed, certainly not the classes of values over which our variables range.

Of course, the notions of dependence and independence apply to quantifiers of any order. Hence we could introduce and also study without any further conceptual cost higher-order IF logics. However, they do not promise any major novelties over and above what we can already examine by reference to first-order IF logic, suitably extended.

One thing that happens in the transition from ordinary ("dependence-handicapped") first-order logic to IF ("hyperclassical") logic is that the semantical structure and the syntactical structure of our formulas are separated from each other. In ordinary first-order logic, the expression within the scope of a quantifier can be thought of as a step in the syntactical formation process of the quantified formula in question. A glimpse at (16)–(17) shows that that is no longer the case in IF logic and in languages whose logic is IF.

This does not show that there is anything wrong with IF languages. What it shows is that what linguists call compositionality does not hold in IF languages. For the basic idea on which the requirement of compositionality is based is that syntactical rules of formation and semantical rules of interpretation operate in tandem. For each rule that tells how an expression F is syntactically built up of its constituent expressions, there is supposed to be a semantical rule to tell how the meaning of F is built up of its constituent expressions. Instead of "meaning", we must here really say more generally "semantical attributes".

This parallelism between syntactical and semantical rules is irretrievably lost in IF logic. The failure of compositionality which has thus been diagnosed, is closely related to the inadequacies of the traditional notion of scope. I have discussed the vagaries of this notion (with an eye also on its uses in linguistics) in an earlier paper (Hintikka [1997]) entitled "No scope for scope?" Its punch line is: "In linguistics, once a day with scope does not do it."

Wilfrid Hodges and his associates have tried to construct a compositional semantics for IF languages. In spite of their great ingenuity, which has led to important new insights into IF logics, the upshot is essentially an impossibility result, as far as my IF logic is concerned. (See Hodges [1997 (a)] and Hodges [1997 (b)] as well as Cameron and Hodges [forthcoming]).

The same loss of compositionality is seen, at least *prima facie*, directly from the intended interpretation of IF logic. The requirement of compositionality is in effect a requirement of semantical context-independence. The semantical attributes of an expression must according to it depend only on the semantical attributes of its component expressions (plus its structure), not on its context. But in IF logic the force of a quantifier depends on its context, viz. on which quantifiers at the outside it depends or does not depend on. This is an unmistakable violation of the principle of compositionality, whose status in logical theory and its applications therefore ought to be reconsidered and deflated.

But if the relaxed parenthesis notation for dependence and independence is impracticable, where do we find a better one? Even more importantly, how can we reach an overview of the different patterns of dependence and independence studied in IF logic? Although we do not have to do so, we are obviously well advised to introduce some new notation. We can for instance express (when needed) the independence of the quantifier $(Q_1 x)$ on another one, say $(Q_2 y)$, by writing it as $(Q_1 x/Q_2 y)$, and its dependence on $(Q_2 y)$ by writing it as $(Q_1 x \backslash Q_2 y)$. (The former is a part of our earlier interim notation.) By means of this notation, we can obviously express any pattern of dependence and independence between quantifiers and *a fortiori* any pattern of dependence and independence between variables. However, purely notationally this still leaves a question open. We still need a way of interpreting unslashed quantifiers. Two different conventions suggest themselves to us here:

(i) Unless otherwise indicated, all quantifiers are independent of each other.
(ii) Unless otherwise indicated, all quantifiers depend on each other.

In spite of their theoretical simplicity, (i)–(ii) lead often to somewhat clumsy formulas. For instance, on the convention (ii), (10)–(11) would be expressed as follows:

(18) $(\forall x/\forall z, \exists y, \exists u)(\forall z/\forall x, \exists y, \exists u)(\exists y/\forall z, \exists u)(\exists u/\forall x, \exists y)\ \mathrm{F}[x, y, z, u]$

(19) $(\forall x)(\forall y)\ \mathrm{F}[x, y]$

On the convention (i), they would be expressed as follows:

(20) $(\forall x)(\forall z)(\exists y\backslash\forall x)(\exists u\backslash\forall z)\ \mathrm{F}[x, y, z, u]$

(21) $(\forall x)(\forall y)(\exists z\backslash\forall y)(\exists u\backslash\forall x)\ ((x = z)\ \&\ (y = u)\ \&\ \mathrm{F}[x, y])$

Neither of these conventions thus yields as perspicuous expressions as would be desirable. Hence an overview is needed here. Instead of changing our notation, we can for this purpose show that each IF formula can be transformed into a normal form logically equivalent with it. This can be thought of as being done in stages. It can be first observed that each formula can be brought to a negation normal formula characterized by the following requirements:

(a) The only propositional connectives used are $\sim$, &, $\vee$.
(b) All negation-signs precede immediately an atomic formula or an identity.

The negation normal form can be reached in the same way as in the received first-order logic. However, we can also bring each first-order formula to what is sometimes called its Skolem normal form, while preserving its negation normal form character. The Skolem form is characterized by the following requirements:

(c) The formula is in a prenex form, that is, it consists of a string of quantifiers (the prefix) followed by a quantifier-free formula.
(d) All universal quantifiers are independent of existential quantifiers.

Explanations: Let us assume that all the relations of dependence and independence among quantifiers are explicitly indicated. Then we can bring a formula to a prenex form in the same way as in the received first-order logic. Moreover, the order of different quantifiers in the prenex form does not matter on the same assumption. Hence we can use the case of adjacent quantifiers as a test case.

The possibility of satisfying (d) can then be seen from the following equivalences:

$$(22) \qquad (\exists x/\forall y)(\forall y\backslash\exists x)\ \mathrm{F}[x, y] \leftrightarrow (\forall y/\exists x)(\exists x/\forall y)\ \mathrm{F}[x, y]$$

$$(23) \qquad (\exists x\backslash\forall y)(\forall y\backslash\exists x)\ \mathrm{F}[x, y] \leftrightarrow (\forall y/\exists x)(\exists x\backslash\forall y)\ \mathrm{F}[x, y]$$

Explanation: Since equivalences are only calculated to preserve truth and since truth means existence of a winning strategy for the verifier, dependence of $(\forall y)$ on $(\exists x)$ does not matter.

Moreover, we can impose on the quantifier prefix the following requirements:

(e) All existential quantifiers are independent of each other.
(f) All universal quantifiers are independent of each other.

Explanation: As with (d), the possibility of satisfying (e) can be seen from the validity of the following equivalences.

$$(24) \quad (\exists x/\exists y)(\exists y\backslash\exists x)\ \mathrm{F}[x, y] \leftrightarrow$$
$$(\forall z/\exists y, \exists x)(\exists x/\exists y, \forall z)(\exists y/\exists x\backslash\forall z)((z \neq x) \vee \mathrm{F}[x, y])$$

$$(25) \quad (\exists x\backslash\exists y)(\exists y\backslash\exists x)\ \mathrm{F}[x,y] \leftrightarrow$$
$$(\forall z/\exists y, \exists x, \forall u)(\forall u/\exists y, \exists x, \forall z)(\exists x/\exists y, \forall z\backslash\forall u)$$
$$(\exists y/\exists x, \forall u\backslash\forall z)((z \neq x) \vee (u \neq y) \vee \mathrm{F}[x, y])$$

Notice that the universal quantifiers introduced in the transition from the left side to the right side in (24) and (25) do not depend on any other quantifiers, even if the replacement happens in context and independently of the other dependence relations of the existential quantifiers in question.

The effects of the transition can perhaps be seen more clearly from the second-order Skolem form of the sentences in question. There you can see that what dependencies between existential quantifiers do is to introduce nesting of the different Skolem functions of a sentence.

Likewise we have the following equivalences for (f):

$$(26) \quad (\forall x/\forall y)(\forall y\backslash\forall x)\ \mathrm{F}[x, y] \leftrightarrow$$
$$(\forall x/\forall y, \exists u)(\forall y/\forall x, \exists u)(\exists u\backslash\forall x/\forall y)((y = u)\ \&\ \mathrm{F}[x, y])$$

$$(27) \quad (\forall x\backslash\forall y)(\forall y\backslash\forall x)\ \mathrm{F}[x, y] \leftrightarrow$$
$$(\forall x/\forall y, \exists z, \exists u)(\forall y/\forall x, \exists z, \exists u)(\exists z/\forall y, \exists u\backslash\forall x)$$
$$(\exists u/\forall x, \exists z\backslash\forall y)((x = u)\ \&\ (y = z)\ \&\ \mathrm{F}[x, y])$$

Actually, it will be assumed that this kind of transformation is carried out in the case of each universal quantifier for all the other universal quantifiers that it depends on in one fell swoop, as indicated by the following equivalence:

$$(28)\quad (\forall x \backslash \forall y_1, \forall y_2, \dots)\, F[x, y_1, y_2, \dots, y_2] \leftrightarrow (\forall x/\forall y_1, \forall y_2, \dots, \exists u)(\forall y_1/\forall x, \exists u)(\forall y_2/\forall x, \exists u)\dots(\exists u \backslash \forall y_1, \forall y_2, \dots/\forall x)((x = u)\ \&\ F[x, y_1, y_2, \dots])$$

Hence only one new existential corresponds to each universal quantifier dependent on other universal quantifiers.

All these equivalences can be verified by expressing their two sides in a second-order form asserting the existence of the appropriate Skolem functions.

The resulting normal form can be written in a more readable form by using the following conventions:

(α) All existential quantifiers are independent of each other.
(β) All universal quantifiers are independent of each other.
(γ) All universal quantifiers are independent of all existential quantifiers.
(δ) An existential quantifier depends on all universal quantifiers unless exempted by the slash notation.
(ε) The formula is in a prenex form with all universal quantifiers preceding all existential ones.

In this notation, the second halves of the equivalences (22)–(28) can be written as follows:

$$(29)\quad (\forall y)(\exists x/\forall y)\, F[x, y]$$

$$(30)\quad (\forall y)(\exists x)\, F[x, y]$$

$$(31)\quad (\forall z)(\exists x/\forall z)(\exists y)(z \neq x \vee F[x, y])$$

$$(32)\quad (\forall z)(\forall u)(\exists x/\forall z)(\exists y/\forall u)((z \neq x) \vee (u \neq y) \vee F[x, y])$$

$$(33)\quad (\forall x)(\forall y)(\exists u/\forall y)((y = u)\ \&\ F[x, y])$$

$$(34)\quad (\forall x)(\forall y)(\exists z/\forall y)(\exists u/\forall x)((x = u)\ \&\ (y = z)\ \&\ F[x, y])$$

$$(35)\quad (\forall x)(\forall y_1)(\forall y_2)\dots(\exists u/\forall x)((x = u)\ \&\ F[x, y_1, y_2, \dots])$$

These examples show how readable formulas in the normal form are. Notice for instance that the reverse slash \ is not needed at all.

Notice also that the quantifier prenex is in the normal form followed by a truth-functional combination of negated and unnegated identities introduced in the transition to the normal form. We can even see the structure of this prefix. In order to eliminate dependencies of universal quantifiers on existential ones, we do not have to introduce any conjuncts or disjuncts. (See (22)–(23).) In order to eliminate dependencies between existential quantifiers we have to introduce disjuncts of the form $(v \neq w)$, where v is bound to a universal quantifier and w to an existential one. (See (24)–(25).) In order to make universal quantifiers independent of each other, we have to introduce identities of the form $(y = u)$ where y is bound to a universal quantifier and u is bound to existential quantifiers. (See (26)–(28).) All told, the prefix thus has the form

(36) $(y_1 = u_1) \,\&\, (y_2 = u_2) \,\&\, \ldots \,\&\, ((v_1 \neq w_2) \vee (v_2 \neq w_2) \vee \ldots \vee \mathrm{F})$

or

(37) $((y_1 = u_1) \,\&\, (y_2 = u_2) \,\&\, \ldots$
$\ldots \,\&\, ((v_1 = w_1) \,\&\, (v_2 = w_2) \,\&\, \ldots) \supset \mathrm{F})$

where F is quantifier free.

It must obviously be understood that this identity prefix is semantically speaking a part of the quantifier prefix. It is not a part of the formula to which quantifiers are prefixed; it is a part of the dependence structure formed by the quantifiers. This fact has to be heeded also in formulating the logical laws of IF logic.

The simplified notation used in the normal form is essentially what I have employed in my earlier work in IF logic. It relies only on the original slash symbol which I introduced earlier as being apparently little more than an interim ad hoc symbolization. Now it turns out to be capable of being able to do the whole notational job needed in IF logic. In the rest of this paper, I will likewise use the same notation unless otherwise indicated. In such a notation, (8)–(9) can be expressed precisely in the same way as in the interim notation

(10) $(\forall x)(\forall z)(\exists y/\forall z)(\exists u/\forall x)\ \mathrm{F}[x, y, z, u]$

(11) $(\forall x)(\forall y)(\exists z/\forall x)(\exists u/\forall y)\ ((x = z) \,\&\, (y = u) \,\&\, \mathrm{F}[x, y])$

It is also of interest to consider a “parametrized” or “time-dependent” version of (11):

(38) $(\forall t)(\forall x)(\forall y)(\exists z/\forall x)(\exists u/\forall y)\ ((x = z) \,\&\, (y = u) \,\&\, \mathrm{F}[t, x, y])$

The normal form has other uses than merely to provide us with a perspicuous notation. Among other things, we can see how different kinds of dependence patterns are expressed in IF logic. We can also come to see how the strategies of the two players of semantical games are represented in IF logic. In order to see this, we can modify the normal form in certain ways. On the basis of what has been said, it is easily seen that the quantifier prefix of a normal form sentence can be split into several sequences of quantifiers of different kinds. More specifically, we have in order the following kinds of quantifier sequences:

(i) A sequence of universal quantifiers $(\forall x)$ such that no identity of the form $(x = u)$ occurs in the identity prefix. Informally speaking, these quantifiers do not depend on anything, but other quantifiers may depend on them.

(ii) Next follows a sequence of universal quantifiers $(\forall x)$ such that an identity $(x = u)$ occurs in the identity prefix. Here u is bound to an existential quantifier of the form $(\exists u/\forall y_1, \forall y_2, \dots)$. These are the universal quantifiers that depend on other universal quantifiers.

(iii) Next follows a sequence of universal quantifiers $(\forall z)$ such that a negated identity $(z \neq w)$ occurs in the identity prefix. Here w is bound to an existential quantifier $(\exists w/-)$ which is followed by $(z \neq w)$ and which occurs within the scope of an existential quantifier $(\exists v/\forall z-)$. Here we can omit $(\forall z)$ as well as $(z \neq w)$ and replace $(\exists w/-)$ by $(\exists w \backslash \exists v/-)$. Informally speaking, such universal quantifiers are merely a means of expressing the dependence of $(\exists w)$ on $(\exists v)$.

Quantifiers of the kind (ii) or (iii) do not occur in traditional first-order logic. Quantifiers of the kind (ii) occur only when mutual dependencies between universal quantifiers are present.

(iv) All the existential quantifiers follow all the universal quantifiers. They are of two kinds. We can assume that first we have the existential quantifiers that have been introduced to specify the dependencies of universal quantifiers on each other, as governed by such equivalences as (26)–(28). There is precisely one $(\exists u)$ such existential quantifier for each universal quantifier $(\forall x)$ of the kind (ii), with the identity prefix containing the identity $(x = u)$.

(v) Each of the remaining existential quantifiers depends on some (or all or none) of the universal quantifiers (i)–(ii).

Next it is of interest to consider a modified form of our normal form. It results through a series of changes. They affect the quantifiers (i)–(v) in different ways.

(i)* Quantifiers of the first kind are not affected.

(ii)* In the case of a quantifier $(\forall x)$ of the second kind, let $(\forall u_1)(\forall u_2), \dots$ be all the universal quantifiers of the second kind other than $(\forall y_1)(\forall y_2), \dots$ Then $(\forall x)$ is replaced by $(\forall x \backslash \forall y_1, \forall y_2, \dots)$. Also, $(\exists u/\forall y_1, \forall y_2, \dots)$ is omitted and so is the identity $(x = u)$.

(iii)* Universal quantifiers of the third kind are omitted. The associated existential quantifiers, $(\exists v/\forall z, -)$ is replaced by $(\exists v/-)$ and $(\exists w/-)$ by $(\exists w \backslash \exists v/-)$. The disjunct $(z \neq w)$ is also omitted.

(iv)* Existential quantifiers of this kind are omitted according to (ii)*.

(v)* Existential quantifiers of this kind may be affected by (iii)*, but not otherwise.

The outcome of this modification results in a sequence of quantifiers without any identity prefix but with dependencies between alike quantifiers (existential quantifiers on other existential ones and universal quantifiers on other universal ones) indicated. I will call it the normal form of the second kind. It brings out especially clearly the dependence and independence relations expressed by the IF sentence in question.

It is also of interest to consider the second-order equivalent of the first normal form. It results from the first normal form (a) by replacing each existentially quantified formula

$$(39) \qquad (\exists z/\forall y_1, \forall y_2, \ldots)\ \mathrm{F}[z]$$

by

$$(40) \qquad \mathrm{F}[f(u_1, u_2, \ldots)]$$

where $(\forall u_1), (\forall u_2), \ldots$ are all the other universal quantifiers than $(\forall y_1), (\forall y_2), \ldots$ and where f is a new function variable different for different formulas (39), and (6) by then prefixing the resulting formula by all the quantifiers $(\exists f)$.

The resulting second-order sentence shows how the strategies of the two players are expressed. The different strategies of the falsifier are represented by all the different sequences of values of the universal quantifiers, restricted by the identities which now take the form

$$(41) \qquad x = f(x_1, z_2, \ldots)$$

where $(\forall z_1), (\forall z_2), \ldots$ are all the universal quantifiers other than $(\forall y_1), (\forall y_2), \ldots$ (cf. (iii) above). These functions f are the ones that replace existentially quantified variables of the kind (iv).

The strategies of the verifier are represented by the functions f that replace existentially quantified variables of the kind (v). The normal forms which have been shown to exist provide us with an overview on IF logic. With the help of this overview, we can formulate various rules of IF logic, not all of which have counterparts in the traditional ("dependence-handicapped") first-order logic. For instance, the negation of a sentence in the negation normal form can be formed simply by exchanging $\exists$ and $\forall$ and by turning negated atomic formulas and identities into unnegated ones, and *vice versa*. Further examples of laws of IF logic are offered by (22)–(27).

By putting such results together, we can among other things show that in a sense in IF logic with exclusively mutually dependent quantifiers a sentence and its negation can in some cases have the same logical form, assuming that the difference between unnegated and negated atomic formulas and identities does not count. It must also be assumed that the pattern of dependence is symmetrical with respect to all the mutually dependent quantifiers. For instance, the negation of (11) can be expressed in the following equivalent forms:

$$(42) \qquad (\exists x)(\exists y)(\forall z/\exists x)(\forall u/\exists y)\ ((x = z)\ \&\ (y = u)\ \&\ \sim \mathrm{F}[x, y])$$

Explanation: the identity conjuncts belong to the quantifier prefix and not to the expression within their scope.

$$(43) \qquad (\forall z)(\forall u)(\exists x/\forall z)(\exists y/\forall u)\ ((x = z)\ \&\ (y = u)\ \&\ \sim \mathrm{F}[x, y])$$

Explanation: compare the Skolem forms of (42) and (43).

$$(44) \qquad (\forall z)(\forall u)(\exists x/\forall z)(\exists y/\forall u)\ ((z = x)\ \&\ (u = y)\ \&\ \sim \mathrm{F}[z, u])$$

Explanation: a trivial consequence of (43). Here (44) and (11) are identical except for the naming of variables and of the negation $\sim$.

By the same token, the negation of the Henkin quantifier sentence (10) is

$$(45) \qquad (\exists x)(\exists z)(\forall y/\exists z)(\forall u/\exists x) \sim \mathrm{F}[x, y, z, u,]$$

This is equivalent with

$$(46) \qquad (\forall y)(\forall u)(\exists x/\forall u)(\exists z/\forall y) \sim \mathrm{F}[x, y, z, u]$$

in other words, equivalent with another Henkin quantifier sentence. This observation throws some light on the claim that GTS does not provide truth-conditions for sentences like (45) (cf. e.g., Tennant [1998]). Such claims are vacuous, for truth-conditions for negations of Henkin quantifier sentences were just seen to have the same form as truth-conditions for unnegated ones.

An interesting feature of irreducibly mutually dependent variables is that one of them cannot be a function of a third variable only. In order to see this, suppose that the Skolem form of (11) (cf. (7)) can be written in the form of

$$(47) \qquad (\exists f)(\exists g)(\forall t)(\forall x)(\forall y)\ ((x = f(t))\ \&\ (y = g(t, x))\ \&\ \mathrm{F}[t, x, y]).$$

But from (40) it can be seen that x does not really depend on y. Hence in the case of irreducibly interdependent quantifiers x, y there cannot for a given value of t in (38) correspond only one pair of values $\langle x, y\rangle$. If t happens to be time, then the values of x and y at some given moment cannot be predicted independently of each other. This shows how at least certain types of indeterminacy phenomena can be dealt with in logical terms.

One predictable but at the same time interesting feature of IF logic is the failure of the usual instantiation rules. In order to see this, consider first what happens if we try to substitute an individual constant, say a, for x in (10). In the result, $(\exists u)$ cannot be independent of such a constant occurring within its scope. Hence the result of this universal instantiation will be

$$(48) \qquad (\forall z)(\exists y/\forall z)(\exists u)\ \mathrm{F}[a, y, z, u],$$

which is equivalent with

$$(49) \qquad (\exists y)(\forall z/\exists y)(\exists u)\ \mathrm{F}[a, y, z, u].$$

But (49) clearly is not implied by (10), even though it is obtained from it by instantiation.

Likewise, the result of substituting a constant, say a, for the first quantified variable in (11) yields according to the same line of thought

$$(50) \qquad (\forall y)(\exists z)(\exists u/\forall y)\ ((a = z)\ \&\ (y = u)\ \&\ \mathrm{F}[a, y]).$$

This is equivalent with

$$(51) \qquad (\exists u)(\exists z)(\forall y/\exists u)\ ((a = z)\ \&\ (y = u)\ \&\ \mathrm{F}[a, y]),$$

which shows that only one value of y is possible here, viz. the one asserted to exist by $(\exists u)$. Hence the multiplicity of possible values of y disappears. According to what has been pointed out, this shows that (51) cannot be implied by (11).

Hence the usual instantiation rules fail in IF first-order logic. How they can be modified so as to become valid requires not only a separate inquiry but further changes in the logic itself.

Other interesting observations can be made here. It is easy to ascertain that neither (10) nor (11) can be reduced in general to a slash-free form even when F is slash-free. For sentences like (11), this brings out another interesting feature of the conceptual situation. Consider as an example a sentence like

$$(52) \qquad (\forall t)(\forall x)(\forall y)\ \ \mathrm{F}[t, x, y]$$

where $\mathrm{F}[x, y]$ is slash-free. It may be thought of as an equation governing certain physical phenomena (but not exhausting them). Suppose furthermore that x and y are mutually dependent. Then (52) has to be replaced by (38). Now this strengthening of (52) cannot be done without the slash notation. For if (38) is irreducibly IF, it cannot be equivalent to a slash-free sentence. Hence (52) cannot be strengthened into (38) without irreducibly informationally independent quantifiers.

One of the most interesting general characteristics of mutually dependent quantifiers is brought out already by our notation. Consider, for the purpose of seeing it, an IF first-order sentence whose universal quantifiers are all interdependent, that is, a sentence without quantifiers of the kind (1) (see above). Such a sentence is of the form

$$(53) \quad (\forall x_1)(\forall x_2)\ldots(\exists z_1/\forall x_1)(\exists z_2/\forall x_2)\ldots \\ ((x_1 = z_1)\ \&\ (x_2 = z_2)\ \&\ \ldots\ \&\ \mathrm{F}[x_1, x_2, \ldots])$$

It is assumed here that some of the $(\exists z_i)$ may be independent of some of the $(\forall x_j)$, but not independent of all of them.

The second-order Skolem form of (46) is

$$(54) \quad (\exists f_1)(\exists f_2)\ldots(\forall x_1)(\forall x_2)\ldots((x_1 = f_1(x_2, x_3, \ldots))\ \& \\ (x_2 = f_2(x_1, x_3, \ldots))\ \&\ \ldots\ \&\ \mathrm{F}[x_1, x_2, \ldots])$$

Here it is instructive to consider a sequence of values of $x_1, x_2, \ldots$ as a vector. Then a full array of Skolem functions $\langle f_1, f_2, \ldots \rangle$ is an operator in the usual mathematical sense of the word: it maps the space of vectors $\langle x_1, x_2, \ldots \rangle$ into itself. I will call it a Skolem operator. The different Skolem operators of a sentence S are from a game-theoretical viewpoint codifications of the different winning strategies that there exist for the verifier in the semantical game $G(S)$ correlated with S. The sentence S is true as soon as there exists a Skolem operator for it. What the Skolem operators of S are, and whether there exist any, depends naturally on the model in which S is being evaluated.

All this can be extended to the general case in which also universal quantifiers of the kind (i) are present by making everything — including Skolem operators depend functionally on those quantifiers.

Skolem operators are thus highly important in logic. One could, and perhaps should, develop the entire theory of first-order logic, IF or not, as a theory of Skolem functions and Skolem operators.

An interesting insight that is facilitated by the Skolem operator viewpoint is seen from (54). How does an IF sentence S with exclusively interdependent quantifiers like (38) represent reality? An answer is seen from (54). *The Skolem vectors that S says are realized in the world* (that is, in the model in which S is evaluated) *are the eigenvectors of its Skolem operators*. The notion of eigenvector assumed here is very nearly the usual mathematical one. The eigenvectors of an operator $O(\langle x_1, x_2, \dots \rangle) = \langle y_1, y_2, \dots \rangle$ are the vectors $\langle a_1, a_2, \dots \rangle$ such that

(55) $$O(\langle a_1, a_2, \dots \rangle) = \langle a_1, a_2, \dots \rangle.$$

If there are independent universal quantifiers present (that is universal quantifiers of the kind (1)), say $(\forall t_1)(\forall t_2), \dots,$ then the situation does not change very much. Instead of (51), we have then

(56) $$(\forall t_1)(\forall t_2)\dots(\forall x_1)(\forall x_2)\dots(\exists z_1/\forall x_1)(\exists z_2/\forall x_2)\dots$$
$$\dots((x_1 = z_1) \,\&\, (x_2 = z_2) \,\&\, \dots \,\&\, \mathrm{F}[t_1, t_2, \dots, x_1, x_2, \dots])$$

and instead of (54) we have

(57) $$(\exists f_1)(\exists f_2)\dots(\forall t_1)(\forall t_2)\dots(\forall x_1)(\forall x_2)\dots$$
$$\dots((x_1 = f_1(t_1, t_2, \dots, x_2, x_3, \dots)) \,\&\, (x_2 = f_2(t_1, t_2, \dots, x_1, x_3, \dots)) \,\&\, \dots$$
$$\dots \,\&\, \mathrm{F}[t_1, t_2, \dots, x_1, x_2, \dots]).$$

Essentially, for any given selection of values of $t_1, t_2, \dots$ the situation is as was described earlier, but with $t_1, t_2, \dots$ as extra parameters in the relevant operators.

In brief, the way in which a proposition with mutually dependent IF quantifiers determines the sequences of values of its quantifiers is the same in which a mathematical equation determines the values of its eigenvectors. Conversely, such eigenvector equations provide the most readily accessible examples of IF sentences with mutually dependent variables.

Straightforward though these observations are, they are highly interesting. In order to see this interest, let us assume that the models of our first-order language have a metric and that the vectors satisfying $\mathrm{F}[x_1, x_2, \dots]$ in (53) form a continuous set. In spite of this continuity, it may happen that the eigenvectors of the Skolem operators of (53), that is, the Skolem vectors admitted by (53), form a discontinuous finite set. Of course, whether this happens depends on $F[x_1, x_2, \dots]$.

The kind of problem situation we face in dealing with mutual dependence is familiar in mathematical physics, where it is sometimes known as quantization. Quantization is an intriguing phenomenon conceptually. How is it that a discontinuous solution can be derived from conditions whose ingredients are all continuous? What has been found out here is that quantization can be a consequence of the mutual dependence of quantifiers. This opens an interesting perspective into the nature of quantization in general.

Most importantly, we can now see that quantization can be dealt with as a purely logical phenomenon, independently of its particular setting in applied mathematics. This extends the range of the potential applications of logic significantly towards the concerns of actual working mathematicians. At the same time these observations serve to illustrate the usefulness of considering the theory of first-order logic as a theory of Skolem functions and Skolem operators.

What has been found here might have more general repercussions than first meets the eye. So far only logic and logical notation has been discussed. Can a mathematical rather than logical notation deal more directly with the problems uncovered by IF logic?

But what can be meant by "mathematical notation" here? Admittedly there is in reality a bewildering variety of mathematical notations. However, the axiomatization and logization of mathematics had the result of showing that much of the usual notation in analysis can be thought of as being at bottom logical. And the relevant logic here includes prominently ordinary first-order logic. One need only consider the Weierstrassian analogues of continuity and differentiation to appreciate this point.

But if so, the characteristic features of IF first-order logic (as distinguished from the old "dependence-handicapped logic") are not captured automatically by the usual mathematical notation, any more than they are captured by ordinary first-order logic. Nevertheless there are plenty of reasons to believe that IF logic has been tacitly used in several parts of mathematics. Now mathematicians are good at creating new notations for their different purposes, or otherwise improvising ad hoc treatments. Hodges [forthcoming] has illustrated the way mathematicians cope with informationally independent quantifiers simply by declaring them to be independent of certain others. Such an ad hoc treatment works in practice, for it is fairly clear intuitively what qualifications such independence imposes on those instantiation rules that bring out the actual force of one's quantifiers. It is for instance relatively easy in building a model for an IF sentence without mutual dependencies to keep track of which choices of new individuals depend on which earlier choices. Barwise [1979] was the first one to point this out.

But no similar tactic works for mutually dependent quantifiers. Even when a mathematician or a physicist becomes somehow aware that certain variables are irreducibly dependent on each other, he or she is not likely to be able to express such dependence in mathematical and logical terms so as to bring out the way in which such a dependence affects the inferences one can draw. For one thing, it was seen earlier that the most familiar vehicles of logical consequence, the usual instantiation rules, become inoperative in the logic of mutually dependent quantifiers. More specifically, in building a model for a sentence with mutually dependent quantifiers it is hard to see how the choices of their values can be correlated. Hence there is no easy way of handling the consequences of mutual dependencies in general without an explicit logical theory. The best one can do is to assert that there are indeed such dependencies, in other words, resort informally to second-order quantification. In mathematical practice, such ascent means replacing variables by functions and functions by operators. But such ascent does not help us to understand what happened to the original independence-free equations and what such an ascent really means. As the comments on the relation of (52) and (38) show, such a step cannot be expressed notationally without the same slash notation whose meaning we are trying to understand.

A beautiful example of an intuitive insight into the relevance of mutual dependence is provided by Paul A. Dirac [1964, p.8], who writes:

(58) Now in the usual dynamical theory, one makes the assumption that momenta are independent functions of the velocities, but that assumption is too restrictive for the applications which we are going to make. We want to allow for the possibility of these momenta *not* being independent functions of the velocities.

It is also instructive to see how Dirac puts his insights into use. He continues:

(59) In that case, there exist certain relations connecting the momentum variables [p] [with the dynamic coordinates q] of the type $\phi(q, p) = 0$.

Here we have a concrete example of the mutual dependence of variables in ordinary discourse — or at least in a practicing physicist's discourse.

It can thus be expected that by making explicit the problems concerning mutually dependent variables IF logic — especially the logic of interdependent quantifiers — can help a mathematician in that it clarifies the conceptual issues involved and thereby shows what logical and mathematical tools he or she should use.

The Dirac example also suggests an answer to a question that the reader might have asked at the outset of this paper. I said there that we must not exclude any possible pattern of dependence and independence from our logic, because it might one day show up in the actual phenomena we are interested in. But do they in fact show up? Do we really need hyperclassical logic for the purposes of any significant applications? The quotation just given provides a clue — no, it provides an example — as to where patterns involving mutual dependence are likely to show up in actual science. (See here Hintikka [forthcoming (a)] and Hintikka [forthcoming (b)].)

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