THE BULLETIN OF SYMBOLIC LOGIC
Volume 8, Number 3, Sept. 2002

ARGUMENTS FOR THE CONTINUITY PRINCIPLE

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Dedicated to the memory of Maurice Boffa (1940–2001)

§1. The continuity principle. There are two principles that lend Brouwer's mathematics the extra power beyond arithmetic. Both are presented in Brouwer's writings with little or no argument. One, the principle of bar induction, will not concern us here. The other, the continuity principle for numbers, occurs for the first time in print in [4]. It is formulated and immediately applied to show that the set of numerical choice sequences is not enumerable. In fact, the idea of the continuity property can be dated fairly precisely, it is to be found in the margin of Brouwer's notes for his course on Pointset Theory of 1915/16. The course was repeated in 1916/17 and he must have inserted his first formulation of the continuity principle in the fall of 1916 as new material right at the beginning of the course.

In modern language, the principle reads

$$\textbf{(WC-N)} \quad \forall\alpha\exists x A(\alpha, x) \Rightarrow \forall\alpha\exists m\exists x\forall\beta\left[\bar{\beta}m = \bar{\alpha}m \rightarrow A(\beta, x)\right]$$

where α and β range over choice sequences of natural numbers, m and x over natural numbers, and $\bar{\alpha}m$ stands for $\langle\alpha(0), \alpha(1), \dots, \alpha(m-1)\rangle$, the initial segment of α of length m.

An immediate consequence of WC-N is that all full functions are continuous, and, as a corollary, that the continuum is unsplittable [28]. Note that WC-N is incompatible with Church's thesis, [22], section 4.6.

After Brouwer asserted WC-N, Troelstra was the first to ask in print for a conceptual motivation, but he remained an exception; most authors followed Brouwer by simply asserting it, cf. [18].

Let us note first that in one particular case the principle is obvious indeed, namely in the case of the lawless sequences. The notion of lawless sequence surfaced fairly late in the history of intuitionism. Kreisel introduced it in [17] for metamathematical purposes. There is a letter from Brouwer to Heyting in which the phenomenon also occurs [7]. This is an important and interesting fact since it is (probably) the only time that Brouwer made use of a possibility expressly stipulated in, e.g., [5], see below.

Received July 16, 2000; revised April 2, 2002.

For Brouwer the universe of choice sequences was of a rich variety, it contained both 'more or less arbitrarily chosen sequences' and lawlike sequences. He envisaged, so to speak, a continuous spectrum of freedom of choice, running from completely free to completely determined. Kreisel, following a suggestion by Gödel, introduced the term 'lawless' for 'completely free'. A lawless sequence is a sequence of natural numbers where at each choice no restrictions are made with respect to future choices. Because so little is known at each moment about such a sequence, one can defend an even stronger principle for them:

$$\textbf{(open data)} \quad A(\alpha) \Rightarrow \exists m \forall \beta \in \bar{\alpha} m A(\beta)$$

where $\beta \in \bar{\alpha}m$ means that β shares the initial segment of length m with α, i.e., $\bar{\alpha}m = \bar{\beta}m$.

Let us just go through the justification for this principle. Suppose we know $A(\alpha)$, i.e., we have a proof for it. Then the proof must be given in a finite time. But at the moment that proof is completed only a finite number of outputs of α have been produced, say, $\alpha(0), \ldots, \alpha(k-1)$. Because α is lawless, the future of α after choice $\alpha(k-1)$ is open, and therefore the proof can only depend on $\bar{\alpha}k$. But then it is clear that A holds for any continuation of $\bar{\alpha}k$. Observe that the same proof applies to $\bar{\beta}k$ $(= \bar{\alpha}k)$, as β is lawless as well. This principle evidently fails for the class of *all* choice sequences, since α could be lawlike, say constant. In that case the principle of open data would not apply to $\exists y \forall x(\alpha(x) = y)$.

So the class of all choice sequences asks for another principle. It is historically interesting that Brouwer avoided the trap suggested by the lawless sequences; his formulation of the continuity principle was right the first time. Hermann Weyl seems to have fallen into the trap, although it must be added that the formulations in Weyl's 'Grundlagenkrise' paper, [29], leave room for other interpretations. These issues are discussed in [25] and [23].

Brouwer's correct formulation of the continuity principle makes it very plausible that he had varying restrictions, restrictions on restrictions, etc., in mind (see below). The later explicit formulation of 'choice sequence' therefore suggests rather a forgotten clause than a new idea.

Can we find a justification for the full principle that runs at least along similar lines as the argument for open data?

Suppose we have a proof of $\forall \alpha \exists x(F(\alpha) = x)$, where F is a function.

Now pick an α and start making the proof instantiation for $\exists x(F(\alpha) = x)$, i.e., we construct a number n and a proof of $F(\alpha) = n$.

A simple minded argument for the continuity principle would be to point out that, when this construction is finished, only finitely many outputs of α have been chosen, and therefore any β with the same initial segment satisfies $F(\beta) = n$.

In order to illustrate the gap in the argument let us consider a simple example.

Imagine the following game between two players, A and B. A generates a choice sequence α and B has a procedure to calculate $F(\alpha)$. For the sake of the example, assume that B will assign to any α the number $\alpha(100)$. Now the game runs as follows: at each move A gives information about α, B looks at it and produces the output (telling A that he may stop), or asks for another bit of information.

Here is a brief run:

A	B
$\alpha(0) = 1$	?
$\alpha(1) = 7$	?
$\alpha(2) = 0$	?
$\alpha(3) = 2$ and	$F(\alpha) = 2$
from now on α is constant.	

So only 4 outputs of α are required to determine $F(\alpha)$. However, it is by no means the case that $F(\beta) = 2$ for any β with initial segment $\langle 1, 7, 0, 2\rangle$.

The error is caused by the fact that A suddenly produces information of a higher order. The argument could therefore be patched up by stipulating that only numerical information may be passed on.

That would, however, not be acceptable at all. It would simply mean that within certain limits (to be precise, the spread conditions) we would act as if all sequences were lawless, and hence establish the principle for that class. But that was not the problem—we wanted to establish the principle for *all* (or maybe a suitable class of) choice sequences.

If we inspect Brouwer's introduction of choice sequences, we see that he realized the difficulties involved in allowing non-lawlike objects. In his first public presentation he hardly paid attention to the sequences, but rather concentrated on laying down the restrictions on numerical choices. This was done in the definition of a *spread* (*Menge*), [4], p. 1:

> A spread is a law on the basis of which, if again and again an arbitrary complex of digits [a natural number] of the sequence ζ [the natural number sequence] is chosen, each of these choices either generates a definite symbol, or nothing, or brings about the inhibition of the process together with the definitive annihilation of its result; for every n, after every uninhibited sequence of $n-1$ choices, at least one complex of digits can be specified that, if chosen as n-th complex of digits, does not bring about the inhibition of the process. Every sequence of symbols generated from the spread in this manner (which therefore is generally not representable in finished form) is called an element of the spread.

> We shall also speak of the common mode of formation of the elements of a spread M as, for short, the spread M.[1]

In modern language, the *spread law* determines a decidable subtree of the tree of all finite sequences of natural numbers satisfying two simple basic conditions:

(i) If $\langle a_0, a_1, a_2, \ldots a_n, a_{n+1}\rangle$ is in the subtree, then so is $\langle a_0, a_1, a_2, \ldots a_n, a_n\rangle$.
(ii) For each finite sequence $\langle a_0, a_1, a_2, \ldots a_n\rangle$ in the subtree, there is an immediate extension $\langle a_0, a_1, a_2, \ldots a_n, a_{n+1}\rangle$ which is also in the subtree.

The second condition ensures that there are no finite maximal paths in the subtree. The infinite paths are called *choice sequences*; they are the *elements* of the spread. No further explanation of the notion of choice sequences is given.

In the next major exposition, [5], Brouwer explicitly allowed over and above the spread condition, restrictions of future choices; in footnote 3 (p. 245) he mentions that he thinks of the elements of a spread as 'including the characteristic of their freedom of continuation, which after each choice can be limited arbitrarily (possibly to complete determination, in any case according to a spread law)'.[2]

In a later, handwritten, note in the margin of a reprint of the same paper, he added

> The *arbitrariness*, subject to the preservation of the possibility of extension, of the 'restriction condition' associated with of a finite choice sequence, lends that choice sequence, and hence also all its extensions, a new arbitrariness. In the spread also an arbitrary well-ordered set of these restriction conditions can be assigned

[1]The original reads: 'Eine Menge ist ein Gesetz, auf Grund dessen, wenn immer wieder ein willkürlicher Ziffernkomplex der Folge ζ gewählt wird, jede dieser Wahlen entweder ein bestimmtes Zeichen, oder nichts erzeugt, oder aber die Hemmung des Prozesses und die definitive Vernichtung seines Resultates herbeiführt, wobei für jedes n nach jeder ungehemmten Folge von $n-1$ Wahlen wenigstens ein Ziffernkomplex angegeben werden kann, der, wenn er als n-ter Ziffernkomplex gewählt wird, nicht die Hemmung des Prozesses herbeiführt. Jede in dieser Weise von der Menge erzeugte Zeichenfolge (welche also im allgemeinen nicht fertig darstellbar ist) heisst ein Element der Menge. Die gemeinsame Entstehungsart der Elemente einer Menge M werden wir ebenfalls kurz als die Menge M bezeichnen.'

[8] contains the following clarifying remark on this definition:

'Because mathematics is independent of language, the word *symbol* (*Zeichen*) and in particular the words *complex of digits* (*Ziffernkomplex*) must be understood in this definition in the sense of *mental symbols*, consisting in previously obtained mathematical concepts.'

[2]'Inklusive des Charakters ihrer Fortsetzbarkeitsfreiheit, welche sich nach jeder Wahl beliebig (eventuell bis zur völligen Bestimmtheit, jedenfalls aber einem Mengengesetze entsprechend) verengern kann.'

(where thus, for example, a restriction of the restrictions for the following choices of a finite choice sequence may be assigned).[3]

In [7] he returned to this particular footnote with the words

> The somewhat brief footnote on the definition of the spread element might gain more clarity in the following more elaborated version:
>
> The freedom of extension of a sequence of series of signs, generated by an unbounded choice sequence, which is an element of the representing spread, can furthermore after each choice be restricted arbitrarily (e.g., to complete determination, or also according to a spread law). And indeed the arbitrariness of the assignment of restriction conditions to individual choices, under preservation of the possibility of extension, forms an essential characteristic of the free growing of the spread element. Each single restriction condition can again be assigned a restriction condition of second order, which restricts the arbitrariness of future restriction conditions, etc.[4]

Hence Brouwer did—at that time—intend to incorporate higher order restrictions. The practical significance of such refinements is not immediately clear. Indeed, in mathematical practice, one usually does not require much more than knowledge of the spread law, and possibly the presence of lawlike sequences. The latter are required to instantiate specific existence claims.

From the viewpoint of metamathematics the lawless sequences are of considerable importance, cf. [15], [22]. They provide the simplest example of a class of choice sequences governed by a higher order restriction, namely the demand that all (first-order) restrictions on the numerical choices will be trivial, i.e., at each stage all numbers are eligible.

For actual mathematics they are, however, not suitable, since they defy even the simplest closure properties. Therefore a more liberal class of choice sequences has to be considered. A general schema for choice sequences looks as follows.

[3]'Die *Beliebigkeit* dieses unter Erhaltung der Fortsetzbarkeitsmöglichkeit einer endlichen Wahlfolge zugeordneten 'Verengerungszusatzes' erteilt dieser Wahlfolge, mithin auch ihren Fortzetzungen eine neue Willkür. Von derartigen Verengerungszusätzen kann nun in der Menge auch eine beliebige wohlgeordnete Spezies angebracht werden (wobei also einer endlichen Wahlfolge z.B. eine Verengerung der für die weiteren Wahlen bestehenden Verengerungszusatzfreiheit zugeordnet werden kann).'

[4]'Die etwas kurz gehaltene Fussnote zur Definition des Mengenelementes dürfte in der folgenden ausführlicheren Fassung an Deutlichkeit gewinnen:

Die Fortsetzbarkeitsfreiheit einer von einer unbegrenzten Wahlfolge erzeugten, ein Element der Menge darstellenden Folge von Zeichenreihen kann übrigens nach jeder Wahl beliebig (z.B. bis zur völligen Bestimmtheit, oder auch einem Mengengesetze entsprechend) verengert werden, und zwar stellt die Beliebigkeit dieser den einzelnen Wahlen unter Erhaltung der Fortsetzbarkeitsmöglichkeit zuzuordnenden Verengerungszusätze einen wesentlichen Charakter des freien Werdens des Mengenelementes dar. Jedem einzelnen Verengerungszusatz kann wieder ein die Beliebigkeit der weiteren Verengerungszusätze einschränkender Verengerungszusatz zweiter Ordnung beigegeben werden, usw.'

The subject successively chooses objects (say natural numbers), restrictions on future choices, restrictions on restrictions of future choices, etc. These lists, of course, have to terminate. So a choice sequence written in full generality looks like

$$\left(n_0, R_0^0, R_1^0, \ldots R_{k_0}^0\right), \left(n_1, R_0^1, R_1^1, \ldots R_{k_1}^1\right), \ldots$$

This agrees with Brouwer's verbal specifications.

Later in life Brouwer started to have qualms about higher order restrictions. In [11], he provided a footnote to the definition of choice sequence, stating that

> In some former publications of the author, restrictions of freedom of future restrictions of freedom, restrictions of restrictions of freedom of future restrictions of freedom, and so on, were also admitted. But at present the author is inclined to think this admission superfluous and perhaps leading to needless complications.

All he allowed was 'Finally the freedom of proceeding, without being completely abolished, may at some p_ν [stage] undergo some restriction, and later on further restrictions.'

In [9], he even went so far as to withdraw his earlier views altogether: 'In former publications I have sometimes admitted restrictions of freedom. However, this admission is not justified by close introspection, and moreover would endanger the simplicity and rigour of further developments.' Heyting, quite sensibly, added the comment that 'It is not clear how introspection could forbid us to introduce the notion of second order restriction. The reason for not introducing them is simply that they are hard to manage.', [10], p. 607.

One might put forward another reason: for the general notion (with full restrictions), it is not clear how to justify the continuity principle. In fact it is not hard to give an example of a class of choice sequences for which the continuity principle does not hold. Consider the full tree of all finite sequences of natural numbers. We will allow choice sequences with first and second order restrictions: there is one constant second-order restriction which says that all first-order restrictions are of the form 'after a finite number of choices the sequence becomes constant'. So each first-order restriction can be written as a finite sequence $(n_0, n_1 \ldots, n_k)$, where $\alpha(0) = n_0, \alpha(1) = n_1, \ldots, \alpha(k) = n_k$, and $\forall p > k(\alpha(p) = n_k)$. This yields perfectly good choice sequences according to the liberal definition. Now we define a function on this set of choice sequences as follows: $F(\alpha) = n_k$, where α is given by the above description. F, clearly, is not continuous!

Note also that the class of all primitive recursive functions falls under the liberal schema of Brouwer.

In spite of the danger of complications, including the loss of the continuity principle, higher order restrictions are useful, not a priori excluded, and perhaps necessary. We wish, however, to retain the continuity principle, the prime instrument of the theory of choice sequences. So, here is a topic for research: find suitable conditions for classes of choice sequences, so that we will be able to use the continuity principle. And, parallel to that, give a justification for the continuity principle that has the widest possible range of application.

§2. A phenomenological consideration.

2.1. An argument for G(raph)WC-N. As Troelstra (e.g., [18], p. 151, [19]) has noted, justifications of WC-N so far have not gone beyond plausibility considerations. In this section, we want to give an informal, rigorous[5] derivation of a *restricted* weak continuity principle. This restricted principle suffices for the intuitionistic reconstruction of analysis, Brouwer's original motivation for the introduction of choice sequences.[6] In the following formulation, the predicate refers only to the *graph* of α.

$$\textbf{(GWC-N)} \quad \forall\alpha\exists x A(\alpha, x) \Rightarrow \forall\alpha\exists m\exists x\forall\beta \left[\bar{\beta}m = \bar{\alpha}m \rightarrow A(\beta, x)\right]$$

Some comments on the condition on A are in order. Troelstra has pointed out (in [20] and in personal communication) that in the case of choice sequences, one can speak of two concepts of extensionality.

First, there is the usual, classical concept, according to which a predicate $A(\alpha, x)$ is extensional if it satisfies

$$\alpha = \beta \wedge x = y \wedge A(\alpha, x) \rightarrow A(\beta, y)$$

Second, there is the stronger concept according to which in the definition of $A(\alpha, x)$, one can only make use of α through its graph. That is to say, α only enters into the formula A through its values.

We will refer to the first concept by it usual name, 'extensionality', and to the second by 'graph-extensionality'.

Clearly graph-extensionality implies extensionality, but the converse does not hold. The reason is that, for incomplete objects, in general the only way to know that $\alpha = \beta$ is to know that they are generated in the very same process. But this opens the door to referring to other information than just the graph. Examples are [20], pp. 221–222, and Visser's proof below, section 2.2, that the present argument does not work for *all* extensional predicates. The latter example explains the restriction on A.

The argument in the present section was arrived at in an attempt to apply Husserl's principle of the noetic-noematic correlation [12], section 93, to choice sequences. Roughly, this principle states that the structure of the way

[5] For the notion of 'informal rigour', see Kreisel [16] and Troelstra [20].

[6] It conforms to the use Brouwer makes of his continuity principle, and also to [28]. The phenomenological details of how the nature of the continuum motivates the introduction of choice sequences are spelled out in [23].

an object is given to us is parallel to the structure of the acts in which that object is intended. In the case of choice sequences, this led to the question: in what ways can the freedom the subject enjoys in the process of generation be reflected in the intensional properties of the sequences themselves? That suggested the idea of provisional restrictions (below), and also the already familiar concepts of lawlike and lawless sequences. It is important to keep in mind that choice sequences are given to us as individual objects.[7]

Let us recall the arbitrariness in posing restrictions, as stressed by Brouwer in the quotes above, see p. 332. This essential freedom motivates the following division of restrictions into two kinds, which we will call 'provisional' and 'definitive'. The subject can make any revision in its ideas about how to go on generating a choice sequence as long as

1. the revision does not go against any other restrictions that are already in effect,
2. the revision admits the existing initial segment,
3. after the revision, it is still possible to extend any admissible initial segment.

Thus we have

definitive restrictions: which have the form 'from now on, restriction R^i_k holds, and it will not be revised anymore' (with or without a specified initial segment);

provisional restrictions: which have the form 'for an unspecified number of stages, restriction R^i_k holds'.

As long as a restriction on a particular sequence is not definitive, that restriction can be reconsidered at some (or any) stage. (Because it is up to the subject's own choice to make a restriction either provisional or definitive, what kind it is dealing with is decidable.) A provisional restriction, in spite of its uncommitted character, is a genuine *restriction*: without lifting it first, one cannot make a choice that does not accord with it—on pain of inconsistent behaviour. That is the informational content of a provisional restriction. Recall our two players A and B from page 331. If A informs B that his next move will not begin by lifting a particular provisional restriction, then B immediately knows something about A's next choice of a value, namely, that it has to respect this unlifted restriction. Had A not imposed this provisional restriction, then B would not have known this.

If several restrictions are in effect, their conjunction counts as 'provisional exactly if at least one of its members is. Note that, for an individual choice sequence to be given to us, it is not required that any of its restrictions is definitive.

[7] Following Troelstra's 'analytic approach', e.g., [19]. See also [23] and [24].

As choice sequences are generated by the subject, imposed restrictions correspond to intensional properties of the sequence. The distinction between provisional and definitive restrictions therefore induces a corresponding distinction between provisional and definitive intensional properties. In phenomenological argot, 'restriction' stresses a *noetic*, 'property' a *noematic* aspect of the constitution of a choice sequence.

On the other hand, not every intensional property corresponds to a restriction. There are other intensional aspects as well, such as the order of generation of the sequence. This is exploited in the arguments in section 2.2.

An example of an already known concept of choice sequence that could be interpreted as an implicit application of the provisional-definitive distinction, is that of the *hesitant sequences* in [22], p. 208:

> A hesitant sequence (say β) is a process of generating values $\beta 0, \beta 1, \beta 2, \dots \in \mathbb{N}$ such that at any stage we either decide that henceforth we are going to conform to a law in determining future values, or, if we have not already decided to conform to a law at an earlier stage, we freely choose a new value of β [...] [T]he decision whether or not to conform to a law may stay open indefinitely.

Thus, we can say that a hesitant sequence is a provisionally lawless sequence (which at any time may be turned into a lawlike sequence).

Let us see how this distinction leads to a justification of GWC-N. We will first consider the case of the universal spread, and then generalize the argument to arbitrary spreads.

The universal spread C is the spread in which each finite sequence $\langle a_0, a_1, a_2, \dots a_n \rangle$ may be extended by any number a_{n+1}. C is the universal tree over $\mathbb{N}$. Since there may be a large variety of conditions (of various orders) on the sequences, we are actually dealing with a universe $\mathcal{U}$ over C.

On p. 253 of [5], Brouwer formulates the following continuity principle for C:

> A law that assigns to each element g of C an element h of A [the natural numbers], must have determined the element h completely after a certain initial segment α of the sequence of numbers of g has become known. But then to every element of C that has α as an initial segment, the same element h of A will be assigned.[8]

The following argument requires that the spread contains, for each element α, the 'provisionalized' version α' as well. 'Provisionalized' means that α and α' have the same initial segments and are subject to the same restrictions, but in the case of α' all of these restrictions are provisional. (Hence α and α'

[8]'Ein Gesetz, das jedem Elemente g von C ein Element h von A [the natural numbers] zuordnet, muß nämlich das Element h vollständig bestimmt haben nach dem Bekanntwerden eines gewissen Anfangssegmentes α der Nummernfolge von g. Dann aber wird jedem Elemente von C, das α als Anfangssegment besitzt, dasselbe Element h von A zugeordnet.'

are not necessarily distinct.) In C, which admits all choice sequences, this requirement is certainly met.

The assignment of a number to a choice sequence requires a construction. At the time of construction, only an initial segment of the sequence is known, as well as some intensional properties. If the intensional properties made use of in the construction are all restrictions, then some of them may turn out to be provisional and therefore open to revision. In this setting, the intensional properties that a sequence has at any particular stage do not necessarily characterize that sequence also at all later stages. That in turn means that a construction that depends on these intensional properties may yield different results at different stages. However, it is understood that once we have a proof that a relation $A(\alpha, x)$ holds between a choice sequence and a number, this proof remains valid once and for all. The problem with a construction that depends on provisional properties is that it cannot guarantee this lasting validity: should the provisional properties change, the outcome of the construction might as well. Therefore such a construction cannot count as a proof of $A(\alpha, x)$.

To illustrate this, let us look at a previous example (p. 331) again. Suppose we have, of a choice sequence α, the initial segment $\langle 1, 7, 0, 2\rangle$, and the *definitive* restriction 'From now on, α is constant'. From this information, we can immediately conclude that $\alpha(100)$ will be 2. Now suppose we have, of a choice sequence β, the initial segment $\langle 1, 7, 0, 2\rangle$, and the *provisional* restriction 'From now on, β is constant'. In this case we *cannot* conclude that $\beta(100)$ will be 2. Maybe we decide to lift the provisional restriction somewhere after our fourth choice; if we do so, we can then consistently choose $\beta(100)$ different from 2.

Earlier on we put the condition on $A(\alpha, x)$ that of α, only the graph is referred to. That allows us to say now that any intensional property that might be useful in constructing an x such that $A(\alpha, x)$, should already be extractable from the first-order restrictions, as they are the only ones where α enters through its values. But we just saw that, if we allow provisionality, restrictions *cannot* be depended on when constructing such an x. Hence the construction can only depend on the other information that we have, given the restriction on A. This other information, of course, is just the initial segment. But that is precisely what GWC-N says.[9]

In [6], p. 63, Brouwer extended the formulation of the continuity principle for C to arbitrary spreads (i.e., subspreads of C):

> Let M be an arbitrary spread, let μ be the denumerably infinite spread of finite (inhibited or uninhibited) choice sequences $F_{sn_1\ldots n_r}$ upon which M is based (where s and the n_ν represent the natural

[9]Note that, because of the condition on A, the presence (or absence) of provisional restrictions on α cannot be exploited in the definition of A.

numbers chosen one after the other for the choice sequence in question), and let a natural number β be associated with each element of M. Then there is distinguished in μ a removable numerable subspread μ_1[10] of uninhibited finite choice sequences such that with an arbitrary element of μ_1 the same natural number β is associated for all elements of M issuing from μ_1, while furthermore a proof h is given that shows, for an arbitrary uninhibited element of μ, that every uninhibited infinite choice sequence issuing from it possesses an [initial] segment belonging to μ_1.[11]

To obtain GWC-N in this case, we resort to a trick. The way to define a subspread in effect consists in specifying which choice sequences are to be removed from C; for some m, their initial segments of length $\geq m$ are inhibited. Hence to these sequences no value will be assigned. But we can embed the subspread into the universal spread if, instead of inhibiting any sequences, we assign a default value to them. (Note that in Brouwer's conception of mathematical entities as generated by the subject, whether or not a sequence is inhibited is decidable.) This embedding enables us to apply the argument for C again, thus showing GWC-N for the subspread as well.[12]

2.2. Two arguments against WC-N. The restriction, in the argument in section 2.1, that A be graph-extensional, is necessary, as shown by the following

[10]I.e., the range of a bijection on $\mathbb{N}$ which is a decidable subset of the total spread [mva and dvd].

[11]'Sei M eine beliebige Menge, μ die ihr zugrunde liegende abzählbar unendliche Menge der endlichen (gehemmten und ungehemmten) Wahlfolgen $F_{sn_1\ldots n_r}$ (wo s und die n_ν die für die betreffende Wahlfolge der Reihe nach gewählten natürlichen Zahlen vorstellen), und sei jedem Elemente von M eine natürliche Zahl β zugeordnet. Alsdann ist in μ eine solche abtrennbare zählbare Teilmenge μ_1 von ungehemmten endlichen Wahlfolgen ausgezeichnet, daß einem beliebigen Elemente von μ_1 für alle aus ihm hervorgehenden Elemente von M dieselbe natürliche Zahl β zugeordnet ist, während weiter eine Beweisführung h vorliegt, mittels welcher sich für ein beliebiges ungehemmtes Element von μ herausstellt, daß jede aus ihm hervorgehende ungehemmte unendliche Wahlfolge einen zu μ_1 gehörigen Abschnitt besitzt.'

The translation is adapted from the English version of [6], p. 63, in [27], p. 459.

[12]The following suggestion is due to Troelstra. It may be possible to model provisional conditions mathematically by projections of lawless sequences, along the following lines. Assume the possible restrictions are enumerable. Think of the generation process of a choice sequence as the choosing of triples $(n_0, R_0, t_0), (n_1, R_1, t_1), (n_2, R_2, t_2), \ldots$. Here, n_i is the number chosen at stage i, R_i is a (provisional) restriction, and t_i is a sequence of integers all $\leq i$ in absolute value. If $+j$ appears in t_i, this means that restriction R_j is made definitive; $-j$ means that restriction R_j is lifted. Observe that after choosing $+j$ one can never at a later stage choose $-j$: definitive restrictions cannot be lifted. In this setting, we may consider the sequence of triples as lawless; what is projected from it is a choice sequence with variable second-order restrictions. No mathematical theory has been developed for those yet. (For the modelling of choice sequences with only (definitive) first-order restrictions by projections, see [21] and [26].)

THEOREM 2.1. *Assume the creating subject generates choice sequences as individual objects, and can therefore enumerate the sequences generated so far. Then WC-N does not hold generally for extensional predicates.*

We have two proofs, the second one actually yielding a stronger result.

FIRST PROOF. Assume we have an operation F (not necessarily a function) that enumerates the choice sequences. Then

$$\forall\alpha\exists n(\alpha = F(n))$$

Put $G(\alpha, n) \equiv \alpha = F(n)$ to get

$$\forall\alpha\exists n G(\alpha, n)$$

and apply WC-N. This yields

$$\forall\alpha\exists m\exists n\forall\beta\left[\bar{\beta}m = \bar{\alpha}m \rightarrow G(\beta, n)\right]$$

But this means that the same n will be paired to different choice sequences, which conflicts with the notion of 'enumeration'. We conclude that in the presence of an enumeration, WC-N does not hold generally. (Note that the predicate G is not graph-extensional.) ⊣

There is also WC-N!, the 'functional' version of WC-N:[13]

$$\textbf{(WC-N!)} \quad \forall\alpha\exists!xA(\alpha, x) \Rightarrow \forall\alpha\exists m\exists!x\forall\beta\left[\bar{\beta}m = \bar{\alpha}m \rightarrow A(\beta, x)\right]$$

The second proof, due to Albert Visser, shows that neither WC-N nor WC-N! holds generally. It is an adaptation of an ingenious trick that Michael Beeson came up with in a recursion-theoretic context [1]. The idea is to construct an extensional predicate based on an intensional property of choice sequences.

SECOND PROOF. We consider an enumeration $(\alpha_n)_n$. Let us introduce some notation:

1. $\boldsymbol{0} = \lambda x.0$, i.e., the constant zero sequence.
2. $P(n) \equiv \bar{\alpha_n}(n) = \bar{\boldsymbol{0}}(n) \vee \exists s < n[\alpha_n(s) \neq 0 \wedge \bar{\alpha_n}(s) = \bar{\boldsymbol{0}}(s) \wedge \exists k \leq s(\bar{\alpha_k}(s+1) = \bar{\alpha_n}(s+1))]$
3. $\mathcal{B}(\alpha_n) = \begin{cases} 0 & \text{if } P(n) \\ 1 & \text{if } \neg P(n) \end{cases}$

Here $\mathcal{B}$ is Beeson's functional, cf. [2], p. 62.

(a) $\mathcal{B}$ is extensional. Assume $\alpha_n = \alpha_m$ and $\mathcal{B}(\alpha_n) = 0$. If the first n values of α_n are 0, then since α_n and α_m are are extensionally equal, the two cases, (i) $m < n$ and (ii) $m > n$, both yield $\mathcal{B}(\alpha_m) = 0$; (i) is obvious, and for (ii) the second part of the clause of $P(m)$ trivially holds.

[13] Similarly, GWC-N has a functional version GWC-N!, which seems to be the principle that Brouwer actually used.

If $\mathcal{B}(\alpha_n) = 0$ on the basis of the second clause of $P(n)$, then there exists an $s < n$ such that $\alpha_n(s) \neq 0$ and $\exists k \leq s(\bar{\alpha_k}(s+1) = \bar{\alpha_n}(s+1))$. Now the two cases are (i) $m < s$ and (ii) $m \geq s$. Both cases immediately give $\mathcal{B}(\alpha_m) = 0$.

(b) $\mathcal{B}$ is discontinuous in $\boldsymbol{0}$.

Consider an initial segment $\bar{\boldsymbol{0}}(k)$ and assume that $\bar{\alpha_n}(k) = \bar{\boldsymbol{0}}(k) \to \mathcal{B}(\alpha_n) = 0$ for all n. Now consider the finitely many α_p with $p \leq k$ and $\bar{\alpha_p}(k) = \bar{\boldsymbol{0}}(k)$, let $a = max\{\alpha_p(k) + 1 | p \leq k \wedge \bar{\alpha_p}(k) = \bar{\boldsymbol{0}}(k)\}$. Pick a $q > a$ such that $\alpha_q(k) = a$ and $\bar{\alpha_q}(k) = \bar{\boldsymbol{0}}(k)$; then by definition $\mathcal{B}(\alpha_q) = 1$.

(c) $\mathcal{B}$ depends on the enumeration.

This can be seen by making a suitable modification of a given enumeration.

Now we are almost done, put $\Phi(\alpha, x) \equiv \exists n(\alpha = \alpha_n \wedge \mathcal{B}(\alpha_n) = x)$. We see that $\forall\alpha\exists x\Phi(\alpha, x)$, and now the first proof shows that weak continuity fails for Φ. ⊣

In view of the crucial role of $\boldsymbol{0}$ a remark seems in order. Wouldn't it be possible that the subject cannot generate the sequence $\boldsymbol{0}$? We know that, for example, in the universe of lawless sequences $\boldsymbol{0}$ does not occur. The answer is that we need not know that the subject can under all circumstances generate the sequence $\boldsymbol{0}$, but that under quite general circumstances he can do so. He can simply say, I make the restriction that from now on I will choose 0. To be precise, he can fix his numerical choices and his restrictions so that $\boldsymbol{0}$ is an admissible sequence.

A brief remark on the possibility to generate a sequence of sequences: the subject can, of course, make only finitely many choices in a finite time, so how can he generate a denumerable sequence of sequences? Here is a method to do so: when generating the sequence $(\alpha_n)_n$, he uses an interleaving process. Step 0: choose $\alpha_0(0)$, step 1: choose $\alpha_0(1)$ and $\alpha_1(0)$, step 2: choose $\alpha_0(2)$, $\alpha_1(1)$, $\alpha_2(0)$, etc.

This second proof is easily generalized to prove

THEOREM 2.2. *For enumerable sets of choice sequences that contain a limit point, neither WC-N nor WC-N*! *holds generally.*

§3. Other arguments for continuity. The following two subsections are proofs of

THEOREM 3.1. *No function* $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *is discontinuous*

3.1. Undecidability of equality of choice sequences. Let f be discontinuous in the sense that a point of positive discontinuity is given. It is no restriction to assume that f is discontinuous in $\boldsymbol{0}$, and that moreover $f(\boldsymbol{0}) = 0$. So we have $\forall n\exists\alpha \in \bar{\boldsymbol{0}}(n)(f(\alpha) \neq 0)$. Let us call the selected α associated to n α_n. Geometrically speaking, we have the leftmost branch $\boldsymbol{0}$ in the underlying tree with a denumerable sequence of side branches α_n, which all produce non-zero outputs under f. It is no restriction to assume that α_{n+1} branches off later than α_n.

The nodes of all the α_n determine a subspread S. We define a mapping g of $\mathbb{N}^{\mathbb{N}}$ onto S by first giving a mapping of the full tree onto the underlying tree T_S of S: each node $\vec{n}$ is taken to the rightmost node $\vec{m}$ of T_S to the left of $\vec{n}$ with the same length. This mapping of nodes induces a unique mapping $g: \mathbb{N}^{\mathbb{N}} \to S$.

We now see that $\forall\alpha(\alpha \# \boldsymbol{0} \leftrightarrow f(g(\alpha)) \neq 0)$, and so $\forall\alpha(\alpha = \boldsymbol{0} \leftrightarrow f(g(\alpha)) = 0)$; hence we get $\forall\alpha(\alpha = \boldsymbol{0} \vee \alpha \neq \boldsymbol{0})$. As the Brouwerian counterexamples show, we cannot expect decidability of equality for choice sequences in a constructive setting. Note that this decidability is called $\forall$-PEM in [22] and Weak limited principle of omniscience (WLPO) in [14]. Our result, plus its (immediate) converse, can be formulated as 'the existence of a discontinuous function from N^N to N is equivalent to WLPO'.

We may also invoke the homogeneity of $\mathbb{N}^{\mathbb{N}}$ and conclude $\forall\alpha\forall\beta(\alpha = \beta \vee \alpha \neq \beta)$, which conflicts with our informal insight.

The fact that our argument dealt with $\mathbb{N}^{\mathbb{N}}$ is not really important, if there were a discontinuous function on a subspread of the universal spread, we could apply the above trick just as well.

3.2. Kripke's Schema and full PEM. There is another approach to the non-existence of a discontinuity; basically it is based on the traditional Brouwerian counterexamples. We will, however, use his later approach of the creating subject. The most convenient way is to employ Kripke's Schema:

$$\textbf{(KS)} \quad \exists\alpha(\exists x(\alpha(x) = 1) \leftrightarrow A)$$

where $\alpha: \mathbb{N} \to \mathbb{N}$ and $\sum_{i=0}^{n} \alpha(i) \leq 1$ for all n.

We consider the same function f as in 3.1, with the same approximating sequences α_n. Moreover, we assume that f is strictly extensional:

$$\textbf{(STREXT)} \quad f(\alpha) \# f(\beta) \to \alpha \# \beta$$

Let α be the Kripke sequence for $P \vee \neg P$ (PEM), i.e., $\exists x(\alpha(x) = 1) \leftrightarrow P \vee \neg P$. Define a sequence δ by

$$\delta(n) = \begin{cases} 0 & \text{if } \forall k \leq n(\alpha(k) = 0) \\ \alpha_k(n) & \text{if } k \leq n \text{ and } \alpha(k) = 1 \end{cases}$$

We evaluate f at δ: $f(\delta) = 0 \vee f(\delta) \neq 0$.

If $f(\delta) = 0$, then $\forall n(\alpha(n) = 0)$, i.e., $\neg(P \vee \neg P)$. Contradiction.

If $f(\delta) \neq 0$, then—because of STREXT—$\delta \# \boldsymbol{0}$. Hence $\delta = \alpha_n$ for some n. And therefore $\exists x(\alpha(x) = 1)$, which implies $P \vee \neg P$. Since P was arbitrary, we get $\forall P(P \vee \neg P)$. It is an intuitionistic credo that not all propositions are decidable, hence there can be no discontinuity of f. More formally: under Kripke's Schema, the existence of a discontinuous strongly extensional function from N^N to N is equivalent to full PEM.

3.3. The KLST theorem. We will show in a number of steps that for strictly extensional functions from choice sequences to natural numbers, the undecidability of equality implies continuity, under assumption of Markov's Principle (MP). The proofs are based on a technique introduced by Ishihara, cf. [13]. This technique goes by the name of 'Ishihara's Trick', see [3]. In fact Ishihara proves his result in a generalized setting. Moreover, he gives necessary and sufficient conditions for continuity, weakening Markov's principle considerably. He considers the general case of complete metric spaces; our proof concentrates on the bare essentials required for metamathematics.

Given the fact from elementary recursion theory that equality between recursive functions is undecidable, the following proof provides a poor man's road to the theorem of Kreisel-Lacombe-Shoenfield-Tseĭtin:

Theorem 3.2. **KLST** (CT,MP) *All functions* $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *are continuous.*

We will split the proof into four lemmas.

Let $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be a non-discontinuous function; and let $\lim_{i\to\infty} \alpha_i = \alpha$. Put $f(\alpha_i) = a_i$, $f(\alpha) = a$. We want to show $\lim a_i = a$. For the topology of the natural numbers $\lim a_i = a$ means just that a_i becomes stationary with value a: $\exists i \forall j > i(a_i = a)$.

We can choose the α_i such that the paths α_i share increasingly longer initial segments with α: there is an increasing sequence N_i such that $k > N_i \to \bar{\alpha}_k i = \bar{\alpha} i$.

$$\begin{array}{ccccccccccc} \alpha_0 & \dots & \alpha_{N_1} & \dots & \alpha_{N_2} & \dots & \alpha_{N_3} & \dots & \to & \alpha \\ \vdots & & \vdots & & \vdots & & \vdots & & & \vdots \\ a_0 & \dots & a_{N_1} & \dots & a_{N_2} & \dots & a_{N_3} & \dots & \stackrel{?}{\to} & a \end{array}$$

Although the behaviour of the a_i is rather mysterious, we know at least something:

Lemma 3.3 (STREXT). $\forall i(a_i = a) \vee \exists i(a_i \neq a)$

Proof. Define $\beta_k = \begin{cases} \alpha & \text{if } \forall j \leq N_k(a_j = a) \\ \alpha_j & \text{else, where } a_j \text{ is the first } a_i \text{ with } a_i \neq a \end{cases}$

It is obvious that (β_k) converges, say to β. Put $f(\beta) = b$. The natural numbers have a decidable equality, so $a = b \vee a \neq b$.

If $a = b$, then, by the definition of β_k, $\forall j(a_j = a)$.

If $a \neq b$, then, by STREXT, $\alpha \# \beta$, and hence $\exists j(a_j \neq a)$. ⊣

Lemma 3.4 (MP, STREXT). $\lim_{i\to\infty} a_i = a$

Proof. The idea is to go up along the path α, checking by Lemma 3.3 if we have reached the point where the a_i's have become constant.

Define the sequence γ_i, and an auxiliary sequence λ_i for applying MP.

1.i If $\exists i(a_i \neq a)$, pick the first a_i with $a_i \neq a$ and put

$$\begin{cases} \gamma_0 = \alpha_i \\ \lambda_0 = 1 \end{cases}$$

1.ii Otherwise, put

$$\begin{cases} \gamma_0 = \alpha \\ \lambda_0 = 0 \end{cases}$$

2 Consider the tail sequence δ_k following α_i: $\delta_k = \alpha_{i+1+k}$. We now apply Lemma 3.3 to (δ_k).

2.i If $\exists i(f(\delta_i) \neq a)$, then put

$$\begin{cases} \gamma_1 = \delta_i & \text{where } i \text{ is the least index with } f(\delta_i) \neq a \\ \lambda_1 = 1 \end{cases}$$

2.ii Otherwise, put

$$\begin{cases} \gamma_1 = \alpha \\ \lambda_1 = 0 \end{cases}$$

3 The general inductive step: let γ_m, λ_m be defined. If $\gamma_m = \alpha$, then $\gamma_{m+1} = \alpha$ and $\lambda_{m+1} = 0$.

Otherwise, we consider the tail sequence δ_k following γ_m. Again, apply Lemma 3.3 to the sequence (δ_k).

3.i If $\exists i(f(\delta_i) \neq a)$, then put

$$\begin{cases} \gamma_{m+1} = \delta_i & \text{where } i \text{ is the least index with } f(\delta_i) \neq a \\ \lambda_{m+1} = 1 \end{cases}$$

3.ii Otherwise, put

$$\begin{cases} \gamma_{m+1} = \alpha \\ \lambda_{m+1} = 0 \end{cases}$$

It is immediate that (γ_i) converges, say to γ. Put $c = f(\gamma)$. Then $c = a \vee c \neq a$.

If $c \neq a$, then, by STREXT, $\gamma \# \alpha$, and hence there cannot be a case (ii), i.e., $\neg\exists i(\lambda_i = 0)$. Therefore $\forall i(\lambda_i = 1)$. But this implies $\gamma = \alpha$. Contradiction.

So $c = a$. If $\forall i(\lambda_i = 1)$, then f is discontinuous (because $f(\gamma) = c \neq f(\gamma_i)$ for all i), so $\neg\forall i(\lambda_i = 1)$, or $\neg\forall i \neg(\lambda_i = 0)$. Therefore $\neg\neg\exists i(\lambda_i = 0)$, and, by MP, $\exists i(\lambda_i = 0)$. This shows that $\lim a_i = a$, and so f is sequentially continuous ⊣

Now we have to make the step from sequentially continuous to continuous. The idea here is to consider a suitable enumerable, dense subset of $\mathbb{N}^{\mathbb{N}}$, and to show that f is continuous on it. By sequential continuity, f is then continuous on the whole set. We fix this enumerable subset by taking all sequences which are eventually 0. They are given by their initial segments preceding the tail of 0's, and hence they can be enumerated; the enumeration is $(\delta_i)_i$. We will prove a lemma, similar to Lemma 3.3.

LEMMA 3.5 (STREXT). *Let f be sequentially continuous. Then*

$$\forall n[\exists\alpha(f(\alpha) \neq n) \vee \forall\alpha(f(\alpha) = n)]$$

PROOF. Consider $\gamma = \lambda i \cdot 1$.

If $f(\gamma) \neq n$, we are done. If $f(\gamma) = n$, we proceed as follows: we consider simultaneously side branches γ_i of γ, where $\gamma_i = \underbrace{1\dots1}_{i}000\dots$, and δ_i's.

(1) If $f(\gamma_0) = f(\delta_0) = n$, we put

$$\beta_0 = \gamma.$$

Otherwise

$$\beta_0 = \gamma_1.$$

(2) Let β_k be defined.

If $\beta_k = \gamma$ then we put

$$\beta_{k+1} = \begin{cases} \gamma & \text{if } f(\gamma_k) = f(\delta_k) = n \\ \gamma_{k+1} & \text{else} \end{cases}$$

And if $\beta_k \# \gamma$, then

$$\beta_{k+1} = \beta_k.$$

Clearly (β_k) converges, say $\lim \beta_k = \beta$. Then $f(\beta) = b$, and $b = n \vee b \neq n$.

If $b \neq n$, then, by STREXT, $\beta \# \gamma$, and hence $\beta = \beta_i$ for some i, so there are γ_j, δ_j with $f(\gamma_j) \neq n$ or $f(\delta_j) \neq n$.

If $b = n$, then $\beta = \gamma$ and $\forall i(f(\delta_i) = n)$. It now immediately follows that $\forall\alpha(f(\alpha) = n)$, as each α is the limit of δ_i's. ⊣

LEMMA 3.6 (MP, STREXT). *f is continuous.*

PROOF. We consider a decreasing sequence of neighbourhoods of some α: $\{\beta|\bar{\beta}i = \bar{\alpha}i\}$, and apply Lemma 3.5 to f on each of these neighbourhoods.

1. If $\exists\beta(f(\beta) \neq f(\alpha))$, then we put $\gamma_0 = \beta, \lambda_0 = 0$ for a β with $f(\beta) \neq f(\alpha)$.
   Otherwise, $\gamma_0 = \alpha$, $\lambda_0 = 1$.
2. If $\exists\beta \in \bar{\alpha}0(f(\beta) \neq f(\alpha))$, then we put $\gamma_1 = \beta$ for a β with $\beta \in \bar{\alpha}0$ and $f(\beta) \neq f(\alpha)$.
   Otherwise, $\gamma_1 = \alpha$.
3. Assume that γ_n has been defined.
   If $\gamma_n = \alpha$, then $\gamma_{n+1} = \alpha$, $\lambda_{n+1} = 1$.
   Otherwise, if $\exists\beta \in \bar{\alpha}n(f(\beta) \neq f(\alpha))$, then we put $\gamma_{n+1} = \beta$ for a β with $\beta \in \bar{\alpha}n$ and $f(\beta) \neq f(\alpha)$, $\lambda_{n+1} = 0$.

Observe that $\lim \gamma_i = \alpha$, and hence, by sequential continuity (lemma 3.4), $\neg\forall i(\lambda_i = 0)$. Therefore $\neg\neg\exists i(\lambda_i = 1)$

By MP, we get $\exists i(\lambda_i = 1)$. Hence $\forall\beta \in \bar{\alpha}i(f(\beta) = f(a))$, i.e., f is continuous. ⊣

§4. Conclusion. Together with the main argument in section 2.1, sections 3.1 and 3.2 seem to exhaust our arguments for continuity. Comparing them, Brouwer's original formulation is the superior one. We note that 3.1 and 3.2 yield only negative results (there are no discontinuities), so Brouwer's continuity principle is appreciably stronger. We may also note that 3.1 and 3.2 are comparable in strength: the undecidability of identity for choice sequences already eliminates discontinuities, but it also rejects full PEM, hence 3.1 subsumes 3.2, not even considering the fact that 3.1 does not depend on KS.

A few more observations may be added: the argument in 3.1 is perfectly general and applies to any notion of choice sequence (say over the universal tree). Therefore it also applies to the lawlike sequences, which are given by pairs $(\alpha(n), R_n)$, where $R_n = R_0$ for all n, and R_0 is a law defining α. Assuming that identity for lawlike sequences is not decidable, one gets 'negative continuity' (i.e., absence of discontinuities) for lawlike sequences. There is no compelling argument for the undecidability of identity for lawlike sequences, although a routine Brouwerian weak counterexample shows that we cannot expect decidability. If one adds Church's Thesis, however, this undecidability can be proved. Brouwer kept clear of continuity for lawlike sequences, but the above arguments show that at least nondiscontinuity can be defended.

Acknowledgments. We wish to thank Andreas Blass, Michael Dummett, Hajime Ishihara, Anne Troelstra, Albert Visser and the referee for their comments, which led to several improvements.

An earlier version of section 2.1 appeared in [23] and was written during a stay at Harvard University in 1998, partially made possible by a grant from the Netherlands Organisation for Scientific Research (NWO), which is gratefully acknowledged.

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