THE BULLETIN OF SYMBOLIC LOGIC
Volume 8, Number 2, June 2002

REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Geoffrey Hellman, Thomas Jech, Wolfram Pohlers, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218). Similar references containing the fraction $\frac{1}{2}$ or a decimal point (such as 70$\frac{1}{2}$***1*** or 3827***.1***) are to *Additions and corrections to A bibliography of symbolic logic* (the JOURNAL, vol. 3, pp. 178–212).

PATRICK BLACKBURN, MAARTEN DE RIJKE, and YDE VENEMA. ***Modal logic.*** Cambridge tracts in theoretical computer science, no. 53. Cambridge University Press, Cambridge, New York, etc., 2001, xxii + 554 pp.

Modal logic is a very broad field, which has demonstrated its usefulness mostly through applications in philosophy, linguistics, and computer science, and occasionally also to mathematics. Yet, the variety of textbooks available for the would-be student is not as rich as in other disciplines of logic. The choice is between Goldblatt, ***Logics of time and computation*** (JSL LVI 1495), Hughes and Cresswell, ***A new introduction to modal logic*** (JSL LXII 1483), and Chagrov and Zakharyaschev, ***Modal logic*** (Clarendon Press, Oxford University Press, 1997). Now Patrick Blackburn, Maarten de Rijke, and Yde Venema, who at one stage worked together in Amsterdam, have added another one to this list. It is a book of seven chapters and more than 500 pages, a book that is highly readable and informative but not without problematic aspects.

It starts with a chapter introducing the basics of modal logic: normal modal logics, consequence relations, frames, models, and general frames. The authors introduce both the plain modal language based on a single operator as well as abstract similarity types, allowing for any number of operators of any arity. Though the book clearly focuses on the basic case (one unary operator), many examples are provided showing the usefulness of the more abstract approach (tense logic, propositional dynamic logic, since and until logic, and

1079-8986/02/0802-0005/$2.30

multidimensional logics). Chapter 2 presents the model theory of modal logic: simulations, bisimulations, the standard translation, the finite model property, modal saturation, ultrafilter extensions, and definability. Chapter 3 continues on that track for frames: it discusses definability, the Goldblatt–Thomason theorem, and then turns to Sahlqvist's theorem, providing a characterization of both the modal axioms as well as the first-order formulas defined by it. In Chapter 4, the book turns to completeness. We learn about canonical models and completeness both via Sahlqvist's theorem and via some kind of tableau calculus. Positive as well as limitative results are given; the extensions of S4.3 are classified according to Bull and Fine, the finite model property of PDL is proved, and we learn about non-canonical logics and about incomplete logics. Chapter 5 provides the key concepts and results of algebraic logic: algebras, general frames, persistence, duality, and the Jónsson–Tarski theorem. After that follows a chapter on complexity of modal logics. After defining the basic notions, some methods for proving decidability and undecidability are given. Then follows a discussion of modal logics that are in NP, in PSPACE, and in EXPTIME. Chapter 7 closes the book. It contains a selection of diverse topics such as since and until logic, hybrid logics, the guarded fragment, and a Lindström-type characterization of modal logic. There are many exercises, there are chapter overviews and various "roadmaps," and three so-called *toolkits*: one on logic, one on algebra, and one on computability. Each chapter also contains a fairly extensive historical and bibliographical overview. Finally, there is a guide to the literature and even a website for the book.

The authors have decided to cover the material that they have worked on themselves, giving the book a bias towards Dutch-style modal logic. Here I think they have greatly succeeded. The exposition of correspondence theory is lucid and presents material that is well worth knowing. Also, the connections with first-order logic are very much up to date. The proofs on preservation classes use saturated models, which greatly simplifies the argument. The connections with computer science are worked out well, and there is substantial relevant material to be found in the book. On the other hand, there are a number of topics that the book does not cover, such as logical deduction, proof methods, interpolation, and intuitionistic logic (to remain with the geographical terminology: there is little Eastern-style modal logic). While the authors acknowledge this in part, I am not convinced by the claim that the book would have grown out of proportion if these topics had been included. For there is other material included that is not of immediate interest for the beginner. The omission of deductive calculi and the proof theory of modal logic severely restricts the usefulness of the book. Even the most basic things either are missing here or else are presented in a very informal manner. The reader does not learn what a rule is let alone an admissible rule, and does not learn about the deduction theorem except in an exercise. This is surprising insofar as the Gabbay-style axiomatization of irreflexivity is discussed, which uses special rules in place of axioms. Moreover, interpolation and related properties are too important to be left unaddressed. It would have cost the authors very little to include these topics and derive valuable results about them; in Kracht, ***Tools and techniques in modal logic*** (BSL VII 278), the reviewer has proposed a method for showing decidability of modal logics that is quite easy to apply and allows one to deduce interpolation for the basic logics in one line. The present arrangement leaves the reader with a somewhat unbalanced view of modal logics. Given this, the novice is well advised to consult Chagrov and Zakharyaschev in addition to the present book, in order to get a complete picture of the field. The survey of the literature that this book contains also suffers from this defect. While the authors praise books of questionable quality, their comments on Chagrov and Zakharyaschev are relatively speaking unfavorable. No mention is made of books that discuss deductive calculi, for example, Rybakov, ***Admissibility of logical inference rules*** (Elsevier, 1997), or Wójcicki, ***Theory of logical calculi*** (JSL LV 1324).

In sum, the book is recommended as an introduction to the more applied modal logic, especially Dutch-style modal logic. But the student who is more interested in the theory of modal logic will find the book too uninformative given that its title suggests that this is a book about modal logic as a whole.

(I have benefited from discussions with Patrick Blackburn, Maarten de Rijke, and Michael Zakharyaschev. The views expressed here are, however, solely my own.)

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MICHAEL ALEKHNOVICH, SAM BUSS, SHLOMO MORAN, and TONIANN PITASSI. *Minimum propositional proof length is NP-hard to linearly approximate*. ***The journal of symbolic logic***, vol. 66 (2001), pp. 171–191.

Let P be a *propositional proof system*, that is, any complete and sound propositional calculus. The traditional proof complexity mostly deals with the question of *existence* of short P-proofs for a given propositional tautology φ. It is, however, quite conceivable (especially for powerful proof systems) that even when short propositional proofs do exist, we nonetheless do not have the slightest clue as to how to find one efficiently. In many cases (and especially from the perspective of automated theorem proving), such a situation is little better than the sheer non-existence of a short proof. M. Bonet, T. Pitassi, and R. Raz in *On interpolation and automatization for Frege systems* (***SIAM journal on computing***, vol. 29 (2000), pp. 1939–1967) proposed a rigorous formulation of what it means to say that *searching* for a short propositional proof does not essentially contribute to the *inherent* complexity of a tautology.

Namely, denote by $S_P(\varphi)$ the minimum bit size of a P-proof of φ. They called a proof system P *automatizable* if there exists a *proof search algorithm* that on an input tautology φ works in time *polynomial in* $S_P(\varphi)$ and outputs a P-proof of φ (such a proof will necessarily be "nearly optimal," that is, its size will also be at most polynomial in $S_P(\varphi)$).

Since the paper under review is entirely devoted to *negative* results about the existence of efficient proof search algorithms, let me briefly summarize what was previously known on the subject.

First of all, we must have some sufficiently strong complexity assumption to start with: without assuming at least $\mathbf{P} \neq \mathbf{NP}$, we cannot rule out the existence of a polynomial time algorithm for any reasonable algorithmic problem, including ours. Then it turns out that the strength of negative results that can be established is determined by two major factors: the strength of this complexity assumption and the strength of the proof system P itself. If we for example assume that certain popular and widely used cryptographic protocols (like RSA) are secure (which is, of course, much stronger than merely $\mathbf{P} \neq \mathbf{NP}$), then really strong proof systems like Frege or extended Frege are known not to be automatizable. If, however, we allow ourselves only $\mathbf{P} \neq \mathbf{NP}$ as the hypothesis (i.e., try to prove ordinary **NP**-hardness results), then our knowledge becomes by far more limited. No propositional proof system is known to be non-automatizable solely under this assumption. Moreover, **NP**-hardness results were known only for the problem of constructing *exactly optimal* proofs (i.e., proofs of size *exactly* $S_P(\varphi)$).

The paper under review proves the first **NP**-hardness results for the problem of constructing efficient *approximate* proof search algorithms. In fact, all its results are applicable in the following more general situation. Note that every efficient algorithm which *produces a P-proof* of nearly optimal size $S_P(\varphi)$, can at the same time be used for approximating the *numerical value* of $S_P(\varphi)$ within the same accuracy. The paper under review rules out efficient algorithms even for the latter, more limited goal of *approximating the minimum propositional proof length*.

Its main result can now be easily formulated in a couple of sentences. Let P be any proof system from the following list: Frege, extended Frege, resolution, Horn resolution, the polynomial calculus, the sequent calculus, or the cut-free sequent calculus (this list comprises quite a number of the most popular systems studied in proof complexity). Then, unless $\mathbf{P} = \mathbf{NP}$, there is no polynomial algorithm for approximating $S_P(\varphi)$ *within a multiplicative factor of* $2^{\log^{1-o(1)}|\varphi|}$; in particular, there is no algorithm approximating $S_P(\varphi)$ *within any given constant factor*.

This result is proved by the reduction technique based on the PCP-theorem which by now has become a standard tool in *computational* complexity. Namely, the authors define another optimization problem MMCSA of computing the minimum weight of a satisfying assignment for a given monotone circuit. Based on the PCP theorem, this problem is shown to be hard even to approximate. Then the authors exhibit polynomial reductions from MMCSA to the problem of approximating $S_P(\varphi)$. What I, however, like most in their paper is the fact that it was the first to exhibit the potential of this standard technique in quite a different context of proof complexity. As the following consideration suggests (and also somewhat explains the slow progress in this area), this should not be easy (and was not in the paper under review).

For any reduction of this sort to work, we must have a sufficient stock of both hard and easy tautologies φ to use in the image of the reduction. Whereas it is generally not a problem with easy ones, possessing hard tautologies should also include a *proof* that they are really hard, and our current abilities for proving lower bounds on $S_P(\varphi)$ (especially not for a single tautology, but for a large class of them, as required by the reduction) are rather limited. This is the major difficulty the authors had to overcome, and they indeed introduced quite a few novel and interesting techniques for this purpose.

As a conclusion, let me mention that it is a widespread belief that for sufficiently strong propositional proof systems, efficient search algorithms which perform reasonably well for *all* tautologies do not exist. The paper under review makes a new significant step toward confirming this belief and paves the way for further progress in this direction.

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LEV D. BEKLEMISHEV. *Induction rules, reflection principles, and provably recursive functions*. ***Annals of pure and applied logic***, vol. 85 (1997), pp. 193–242.

According to a classical result by Kreisel and Lévy (*Reflection principles and their use for establishing the complexity of axiomatic systems*, JSL XXXVI 529), the uniform reflection principle for EA (also known as $\mathrm{I}\Delta_0 + \mathrm{EXP}$) and the schema of induction are equivalent over EA. Therefore the *rule* of induction and the uniform reflection principle for EA are also equivalent, simply because the rule of induction and the schema of induction are equivalent.

Now one can ask how much reflection is needed for how much induction. Leivant (*The optimality of induction as an axiomatization of arithmetic*, ***The journal of symbolic logic***, vol. 48 (1983), pp. 182–184) has refined Kreisel's and Lévy's result by investigating restricted schemata of induction; he proved that $\mathrm{I}\Sigma_k$ is equivalent to EA plus uniform reflection for Σ_{k+1}-formulas, if $k \geqq 1$.

Beklemishev takes up the task of relating reflection schemata and restricted induction *rules*. He considers not only restrictions on the complexity of the formulas allowed in the rule of induction but also restrictions on the number of allowed iterations of the rules. In particular, he considers for theories T and rules R the closure $[T, R]$ of T under uniterated applications of R. That is, a proof in $[T, R]$ may contain several applications of R, but they are not to occur on the same branch within the proof.

Beklemishev puts also a strong emphasis on the base theories used. These careful distinctions help to clarify the relationship of the different axioms and rules: If n iterations of a rule correspond to n nested applications of another rule, this indicates a close relation between the two rules. Also their relationship is stronger if the equivalence obtains over a great variety of base theories. All theories considered are formulated in arithmetical languages.

If the base theory is fixed, then one can formulate reflection principles by explicitly producing a suitable provability predicate. But there is no canonical representation of arbitrary axiom sets and thus no canonical provability predicate for any arbitrarily given theory. For theories with finite axiom sets, however, there obviously are canonical provability predicates (see Feferman's *Arithmetization of metamathematics in a general setting*, JSL XXXI 269). Thus general considerations without fixed theories usually proceed in terms of reflection principles for theories with finite axiom sets. Beklemishev's paper is no exception.

His analysis of Π_n-IR is as follows. If $n \geqq 1$, T is a theory containing EA (or in the case $n = 1$ the theory $\mathrm{I}\Delta_0 + \mathrm{SUPEXP}$ because $[T, \Pi_n\text{-IR}]$ must prove the formalized cut-elimination theorem), then $[T, \Pi_n\text{-IR}]$ is equivalent to T together with the uniform reflection principle

$$\forall x(\mathrm{Bew}_{T_0}(\ulcorner \varphi(\dot{x}) \urcorner) \rightarrow \varphi(x)) \tag{1}$$

for all finite Π_{n+1} axiomatized theories $T_0 \subseteq T$.

Although Σ_n-IR is known to be equivalent to Π_{n+1}-IR over EA, the above result cannot be easily transformed into an analysis of Σ_n-IR because T with the uniterated rules, i.e., $[T, \Sigma_n\text{-IR}]$ and $[T, \Pi_{n+1}\text{-IR}]$, may be different theories. Thus the following result cannot be obtained directly. If T contains $\mathrm{I}\Sigma_n$, then $[T, \Sigma_{n+1}\text{-IR}]$ is equivalent to T together with the uniform reflection principle (1) for all finite Π_{n+2} axiomatized theories $T_0 \subseteq T$. Thus at least for Π_{n+2} axiomatized theories containing $\mathrm{I}\Sigma_n$, $[T, \Sigma_{n+1}\text{-IR}]$ and $[T, \Pi_{n+2}\text{-IR}]$ are the same theories.

These results imply the following characterization of the induction rules in terms of iterated reflection principles: EA plus Σ_n-IR (or Π_{n+1}-IR) is equivalent to the ω-times iterated reflection principle for Σ_n formulas (starting from EA).

Beklemishev's paper is quite readable and is accessible even to readers not familiar with this area of proof theory. He introduces the basic notions carefully and reviews some well-known results in order to set the stage for his own work. The paper also contains some new proofs of old results and thereby provides new insights. The proofs are worked out in detail.

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Uri Abraham, Matatyahu Rubin, and Saharon Shelah. *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types.* ***Annals of pure and applied logic***, vol. 29 (1985), pp. 123–206.

There may have been a time when it seemed that any problem related to the countable chain condition (ccc) would eventually be solved by Martin's axiom. If there was, it surely ended with the appearance of the paper under review. As the authors point out in their introduction, much of the impetus for the direction of research reported in this paper derived from the following result of Baumgartner generalizing Cantor's classic theorem that every two dense linear orders are isomorphic: It is consistent that any two $\aleph_1$-dense subsets of $\mathbb{R}$ are order-isomorphic. (A set of reals is said to be $\aleph_1$-dense provided that its intersection with each rational interval has size $\aleph_1$.) Since Baumgartner's argument relied on a finite-support iteration of ccc partial orders, as well as a great deal of cleverness, there was interest in seeing whether at least the cleverness could be dispensed with and the result obtained as a direct consequence of Martin's axiom. This, however, was shown to be impossible by Shelah in 1981; he showed it to be consistent with Martin's axiom that there is a set of reals, known as a *k-entangled set*, which has the property that, given any uncountable family $\{(a^1_\alpha, a^2_\alpha, \dots, a^k_\alpha)\}_{\alpha\in\omega_1}$ of k-tuples from this set in which no real appears more than once, every element of $\{-1, +1\}^k$ occurs as

$$\left(\frac{a^1_\alpha - a^1_\beta}{|a^1_\alpha - a^1_\beta|}, \frac{a^2_\alpha - a^2_\beta}{|a^2_\alpha - a^2_\beta|}, \dots, \frac{a^k_\alpha - a^k_\beta}{|a^k_\alpha - a^k_\beta|}\right)$$

for some $\alpha < \beta$. In particular, the set is not isomorphic to its reflection and, since it can easily be guaranteed to be $\aleph_1$-dense, this shows that Martin's axiom does not suffice for Baumgartner's result. The key new ingredient to this construction was that the entangled set can not be chosen to be ccc-indestructible, as was the case in earlier arguments such as Szentmiklóssy's ccc-indestructible S-space. Indeed it is possible that during the iteration to obtain Martin's axiom, ccc partial orders destroying the entangled set will be encountered; but these are dealt with by adding uncountable antichains to them rather than generic sets. Although embryonic forms of such strategies can be found in Jensen's argument showing that Suslin's hypothesis does not contradict the continuum hypothesis, the paper under review provides a general framework for using this idea as well as other strategies for achieving the same ends.

The paper devotes considerable attention to a collection of forcing schemes that can be used to establish not only Baumgartner's result but a number of generalizations as well. Perhaps the most significant contribution of the paper, though, is the distillation of these techniques into a family of partition axioms which the authors then use to establish further results. The simplest of the infinite partition statements, at least for pairs, is Ramsey's theorem, denoted by

$$\aleph_0 \longrightarrow (\aleph_0)^2_2$$

which says that, for any colouring of the pairs of a countable set into two colours, there must be an infinite subset all of whose pairs are coloured the same. Already in the early part of the last century, Sierpiński had shown that this can not be generalized to sets of size $\aleph_1$; in other words, the relation

$$\aleph_1 \longrightarrow (\aleph_1)^2_2$$

is false. The key observation made by Abraham, Rubin, and Shelah is that if one restricts the class of colourings to have some nice properties, then positive generalizations of Ramsey's theorem are possible in spite of Sierpiński's counterexample. The context they use to define "nice" is the topology of the real line. If $X \subseteq \mathbb{R}$ and $f : [X]^2 \to 2$, then Abraham, Rubin, and Shelah define f to be semi-open if $f^{-1}\{1\}$ is open. They define the *semi-open coloring axiom* (SOCA) to be the assertion that for every $X \in [\mathbb{R}]^{\aleph_1}$ and every semi-open $f : [X]^2 \to 2$ there is $Y \in [X]^{\aleph_1}$ that is monochrome in the sense that f is constant on $[Y]^2$. One might imagine that there might now be an entire hierarchy whose strength depends on the topological complexity one allows the colourings. It turns out, though, that as soon as one allows colourings even one step higher in the Baire hierarchy than semi-open, the resulting axiom is inconsistent.

Nevertheless, SOCA can be put to various uses and has, since the publication of this paper, established itself in a central position in the development of the combinatorics associated with the reals. For example, Kunen's technique of freezing Hausdorff gaps to make them absolute (see *Where MA first fails*, ***The journal of symbolic logic***, vol. 53 (1988), pp. 429–433) can easily be formulated into an application of SOCA. This has played a large part in dealing with problems about automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$. Questions related to the structure of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$ and of $({}^\omega\omega, \leq^*)$ also lend themselves to such an approach. In the paper under review, however, the authors focus their attention on applications of SOCA such as the following: Every partial, uncountable function from $\mathbb{R}$ to $\mathbb{R}$ is monotone on an uncountable set.

A remark on the subsequent development of SOCA may be in order here, since it is possible that some readers may be questioning the significance attributed to SOCA, claiming they have never even heard of it. Abraham, Rubin, and Shelah were able to extend their results on SOCA to show that it is consistent with Martin's axiom that the following holds: Given a set of reals X of size $\aleph_1$ and a semi-open $f : [X]^2 \to 2$, either there is an uncountable monochrome set in the open colour, 1, or X is the union of countably many monochrome sets (without loss of generality of colour 0). An important rôle in the argument establishing this consistency is played by the following observation: If f is semi-open and f is constant with value 0 on $[A]^2$, then the same is true for the closure of A. This allows arbitrary monochrome sets in the closed colour to be replaced by closed ones and, thus, candidates for countable families of monochrome sets can be enumerated in length $2^{\aleph_0}$. Moreover, absoluteness can also be applied and Todorcevic was later able to exploit this to establish the consistency of the following axiom: For every set of reals X (no restriction on its size) and for every semi-open $f : [X]^2 \to 2$, either there is $Y \in [X]^{\aleph_1}$ such that f is constant with value 1 on $[Y]^2$ or X is the union of countable many monochrome sets. (See Chapter 8 of ***Partition problems in topology***, JSL LVI 1488.) He obtained several applications of it and christened the resulting axiom OCA. The applications of OCA that use its full strength are those dealing with structures of size greater than $\aleph_1$; for those that claim OCA as a hypothesis but deal only with structures of size $\aleph_1$, SOCA actually suffices.

To make matters even more confusing, the paper under review also contains an axiom called OCA which has little to do with what is currently known as OCA. The original OCA says the following: For any $X \subseteq \mathbb{R}$ of size $\aleph_1$ and any continuous colouring $f : [X]^2 \to n$ of the pairs of X with finitely many colours, there are monochrome sets $\{A_i\}_{i\in\omega}$ such that $X = \bigcup_{i\in\omega} A_i$. The key application of OCA provided by Abraham, Rubin, and Shelah is to the structure of the family of all $\aleph_1$-dense sets under the relation $\prec$ of order embedding, which they call K. They also consider the substructure $K^H \subseteq K$ consisting of the family of homogeneous $\aleph_1$-dense sets. They show that under Martin's axiom, OCA implies that the K and K^H are very simple: Either every two $\aleph_1$-dense sets are isomorphic or there are three homogeneous $\aleph_1$-dense sets such that every homogeneous $\aleph_1$-dense set is isomorphic to one of them. They go on to show that if one includes the involution operation $*$ (which sends x to $-x$) then the structure $(K^H, \prec, *)$ can consistently be any finite distributive lattice with involution. This result does not seem to have received the attention one might have expected, perhaps because it seems to have completely solved the problem. Nevertheless, the question of what infinite lattices are possible still seems to be open.

Finally, it is worth mentioning that Abraham, Rubin, and Shelah show that SOCA is consistent with the continuum's being larger than $\aleph_2$; but this still remains an open question for Todorcevic's OCA and, purely coincidentally, is also open for the original OCA as well. (Recently, Justin Moore has, however, shown that the conjunction of the two versions of OCA does imply that $2^{\aleph_0} = \aleph_2$; see *Open colorings, the continuum and the second uncountable cardinal*, ***Proceedings of the American Mathematical Society***, posted 12 February 2002, PII S 0002-9939(02)06376-1.) In the same spirit, the following question of Baumgartner, which served as a catalyst for the paper under review also remains open: Is it consistent that every two $\aleph_2$-dense sets are isomorphic?

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Z. Szentmiklóssy. *S-spaces and L-spaces under Martin's axiom*. ***Topology***, Volume II, edited by A. Császár, Colloquia mathematica Societatis János Bolyai, no. 23, János Bolyai Mathematical Society, Budapest, and North-Holland Publishing Company, Amsterdam, Oxford, and New York, 1980, pp. 1139–1145.

Zoltán Balogh. *On compact Hausdorff spaces of countable tightness*. ***Proceedings of the American Mathematical Society***, vol. 105 (1989), pp. 755–764.

The papers under review fall under applications of set-theoretic methods in topology. In both cases the logical component of these methods is essential; in the case of the first paper it is Martin's axiom and in the case of the second paper under review it is the proper forcing axiom. One can identify two flourishing periods in the development of consistency results in set-theoretic topology, related to applications of the two above-mentioned axioms respectively. The papers can be considered as some of the leading representatives of these periods. In the meantime, extensive and excellent literature has appeared that deals with the results of the papers.

The main result of the paper of Z. Szentmiklóssy is that under the assumption of MA + $\neg$ CH there are no compact S-spaces. MA stands for Martin's axiom which is an independent (in the logical sense) set-theoretic statement about partial orders that satisfy the countable chain condition (c.c.c.). It can be used in a variety of ways in mathematical constructions that can be approximated by a partial order. CH stands for the continuum hypothesis. S-spaces are hereditarily separable but not hereditarily Lindelöf regular spaces and L-spaces are hereditarily Lindelöf but not hereditarily separable regular spaces. These two classes of topological spaces are pretty natural, since separability and the Lindelöf property are equivalent in the class of metrizable spaces. The first examples of Hausdorff S-spaces and L-spaces, without requiring regularity, were obtained without any special set-theoretic assumptions by Wacław Sierpiński in *Sur l'équivalence de trois propriétés des ensembles abstraits*, ***Fundamenta mathematicae***, vol. 2 (1921), pp. 179–188. Nevertheless, regular (and first countable and compact) examples were obtained only under various additional set-theoretic assumptions, first by assuming CH in A. Hajnal and I. Juhász, *On hereditarily α-Lindelöf and α-separable spaces, II*, ***Fundamenta mathematicae***, vol. 81 (1974), pp. 148–158. Thus, the main result of the paper, that the usual axioms of set theory ZFC are not sufficient to construct a compact S-space, is very interesting.

In fact, the author proves stronger results which are based on a lemma in ZFC which has become known as Szentmiklóssy's lemma. (See J. Roitman, *Basic S and L*, JSL LII 1044. Roitman devotes almost the entire Section 6 to the results of the paper under review.) The arguments of the paper were later strengthened and dualized. (See for example U. Abraham and S. Todorcevic, *Martin's axiom and first-countable S- and L-spaces*, JSL LII 1044.)

In the early eighties, S. Todorcevic pioneered the use of proper forcings (a notion due to S. Shelah) to obtain undecidability of topological problems. Actually, he introduced a general approach, called *models as side conditions*, which describes how to redesign a forcing notion (a partial order used to obtain a generic extension of a model of ZFC) that fails to be c.c.c. (and thus MA cannot be applied) and obtain a proper forcing notion that may do the job of producing a consistency proof. Using this approach, among others, he solved a long-standing open problem strengthening Szentmiklóssy's result: One cannot construct even a regular S-space without special set-theoretic assumptions; the proper forcing axiom implies that there are no S-spaces. The present state of the art on S- and L-spaces, including the method of models as side conditions, is described by Todorcevic in ***Partition problems in topology*** (JSL LVI 1488). This book gives the combinatorial essence of Szentmiklóssy's result in Theorem 7.10 (see also interesting historical remarks about Szentmiklóssy's result on page 65).

Balogh's paper (the second under review) is a daring adaptation of Todorcevic's method in the context of countably compact spaces, employing also Fremlin's and Nyikos's ideas. The consequences are of historical importance.

The most important is the solution of the Moore–Mrówka problem whether countably tight compact spaces are sequential. Balogh proves that the answer *yes* is consistent. Countable tightness and sequentiality are among the most important generalizations of the property of having a countable base at each point, and play an important role in the theory of non-metizable function spaces. A space has countable tightness if the fact that a point belongs to the closure of a subspace A means that it belongs to the closure of some countable subset of A. A space is sequential if for being a closed subspace it is enough to be closed under limits of convergent sequences. (Remember that such a nice classical compact space as βN does not have any non-trivial convergent sequences; it is neither countably tight nor sequential.) Several other important consequences are obtained, for example, a countably compact manifold is metrizable if and only if it does not contain a copy of ω_1.

The complementary constructions of countably tight compact spaces that are not sequential or even have no converging sequences are due to Ostaszewski and Fedorchuk respectively. They belong to the earlier period in the history of set-theoretic topology and were obtained under the assumption of $\diamondsuit$.

The link between Todorcevic's original method and Balogh's proof was later analyzed by Alan Dow in *An introduction to applications of elementary submodels to topology* (BSL VII 537), where one more fruit was harvested: quoting the above paper, "just something that Fremlin, Nyikos and Balogh 'missed' " is that PFA implies that each countably tight compact space has a point with a countable neighborhood base. Balogh's methods also are related to research of other mathematicians most of them are coauthors of the paper by Z. Balogh, A. Dow, D. Fremlin, and P. Nyikos, *Countable tightness and proper forcing*, ***Bulletin of the American Mathematical Society***, n.s. vol. 19 (1988), pp. 295–298. In this paper the authors also note that the assumption of the existence of large cardinals used in the second paper under review is not necessary to obtain its results.

The fact that the Moore–Mrówka problem is undecidable nowadays belongs to the canons of independence in mathematics. Understanding the results of the papers is "a must" for anyone seriously interested in the use of forcing in topology. Two famous open problems intimately related to the results of the papers under review are probably waiting for presently unavailable forcing methods: Is it consistent that there are no L-spaces? Is it consistent that if X is compact then it contains either a copy of βN or a copy of a convergent sequence (Efimov)? Therefore the papers or the study of the results of the papers (now available in other sources mentioned above) could also be suggested to all logicians and set-theorists who are developing new forcing methods.

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Maxim R. Burke and Menachem Magidor. *Shelah's pcf theory and its applications*. ***Annals of pure and applied logic***, vol. 50 (1990), pp. 207–254.

The paper under review was published before Shelah's book on pcf theory (***Cardinal arithmetic***, JSL LXII 1035) appeared, with the authors' expressed "hope that what we do present will serve as an accessible introduction to the original papers" (p. 208). In fact the Burke–Magidor paper continued to be a frequently quoted source for pcf theory in the years after the appearance of ***Cardinal arithmetic***, because Shelah's style of writing made his book difficult to read for many.

This paper presents the fundamental theorems of pcf theory: the λ-directedness of $(\prod A/J_{<\lambda})$, the *convexity theorem* (that pcf of a small interval of regular cardinals is an interval of regular cardinals), the existence of pcf generators, and localization. Applications of pcf theory are presented also. Within just a few pages from the beginning, the existence of a Jónsson algebra on $\aleph_{\omega+1}$ is proved. Results in the partition calculus are also presented.

The existence of pcf generators for a set A of regular cardinals is first proved under the assumption that $2^{|A|} < \min A$ and then with the standard assumption $|A| < \min A$, but the existence of transitive generators, and consequently localization and the equation $\mathrm{PCF}\,\mathrm{PCF}\,A = \mathrm{PCF}\,A$, are proved only under assumptions that are not optimal.

The style of presentation is more that of lecture-notes than of a survey paper. Apart from the repetition in having two proofs for the existence of generators, there are instances where definitions are made within the statement of a lemma, and some proofs are not quite polished.

PCF theory has continued to develop rapidly in the 1990's and therefore many important notions—such as that of a good scale—and important theorems—such as the asymptotic version of GCH in ZFC (Saharon Shelah, *The generalized continuum hypothesis revisited*, ***Israel journal of mathematics***, vol. 116 (2000), pp. 285–321) and its consequence that for sufficiently large λ, $\diamondsuit_{\lambda^+}$ holds iff $2^\lambda = \lambda^+$—of course are not covered by this paper. The *Analytical guide* and the *Annotated content of continuations* sections at the end of ***Cardinal arithmetic*** are recommended to readers who wish to check what is known in the field (or rather, what was known about 1994). Another newer source of pcf theory is Holz, Steffens, and Weitz, ***Introduction to cardinal arithmetic*** (Birkhäuser, 1999), which is partially based on the paper under review and which contains additional material.

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THOMAS JECH. *Singular cardinal problem: Shelah's theorem on $2^{\aleph_\omega}$*. ***Bulletin of the London Mathematical Society***, vol. 24 (1992), pp. 127–139.

In this paper Tom Jech presents, in his familiar readable style, a self-contained proof in about ten pages of Shelah's

THEOREM 1. $[(\forall n)(2^{\aleph_n} < \aleph_\omega)] \Rightarrow 2^{\aleph_\omega} < \aleph_{\omega_4}$

The secret of the paper's brevity is the repeated use it makes of the assumption $2^{\aleph_0} < \aleph_\omega$ in proving particular instances of pcf theorems—theorems that hold also *without* this assumption, indeed without any assumptions at all beyond ZFC. The use of the additional assumptions allows a considerable simplification of the proofs (for example, the existence of least upper bounds modulo ideals can be gotten from the Erdős–Rado theorem easily, but needs a lengthy argument otherwise).

Shelah proved in fact a stronger theorem:

THEOREM 2. $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4}$

This theorem is stronger because it makes no assumptions on the size of the continuum; in the (consistent) case that the continuum is equal to $\aleph_{\omega_5+23}$, it is still true, by this theorem, that the least number of countable subsets of $\aleph_\omega$ required to cover *all* countable subsets of $\aleph_\omega$ is smaller than $\aleph_{\omega_4}$, although each countable set contains more than $\aleph_{\omega_4}$ subsets.

Shelah's work on cardinal arithmetic made it clear that in ZFC itself, many good theorems about *revised* singular cardinal arithmetic can be proved—such as the one above, which gives an absolute upper bound on the *cofinality* of $[\aleph_\omega]^{\aleph_0}$, rather than on its cardinality (which would be of course impossible, by Cohen's proof). So in this respect there is a major point lost in the presentation made in this paper, and it may be a bit misleading in not emphasizing this innovative aspect of Shelah's contribution to cardinal arithmetic.

In this paper the author does a great job in presenting the first theorem above in minimum space; but a reader who would like to know why there exists a Jónsson algebra on $\aleph_{\omega+1}$—or why the *second* theorem above is true—would not be able to use any of the proofs in it.

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YOSHIHIRO ABE. *Weakly normal filters and the closed unbounded filter on $P_\kappa\lambda$*. ***Proceedings of the American Mathematical Society***, vol. 104 (1988), pp. 1226–1234.

YOSHIHIRO ABE. *Weakly normal filters and large cardinals*. ***Tsukuba journal of mathematics***, vol. 16 (1992), pp. 487–494.

YOSHIHIRO ABE. *Weakly normal ideals on $\mathcal{P}_\kappa\lambda$ and the singular cardinal hypothesis*. ***Fundamenta mathematicae***, vol. 143 (1993), pp. 97–106.

YOSHIHIRO ABE. *Saturation of fundamental ideals on $\mathcal{P}_\kappa\lambda$*. ***Journal of the Mathematical Society of Japan***, vol. 48 (1996), pp. 511–524.

YOSHIHIRO ABE. *Strongly normal ideals on $\mathcal{P}_\kappa\lambda$ and the Sup-function*. ***Topology and its applications***, vol. 74 (1996), pp. 97–107.

YOSHIHIRO ABE. *Combinatorics for small ideals on $\mathcal{P}_\kappa\lambda$*. ***Mathematical logic quarterly***, vol. 43 (1997), pp. 541–549.

YOSHIHIRO ABE and MASAHIRO SHIOYA. *Regularity of ultrafilters and fixed points of elementary embeddings*. ***Tsukuba journal of mathematics***, vol. 22 (1998), pp. 31–37.

Throughout this review, κ will denote a regular uncountable cardinal and λ a cardinal $> \kappa$. $P_\kappa\lambda$ is the collection of all subsets of λ of size $< \kappa$. As a convenient convention, the phrase "ideal on $P_\kappa\lambda$" will mean "proper, κ-complete, fine ideal on $P_\kappa\lambda$."

Since the formulation of the concept of supercompactness by Reinhardt and Solovay in the late sixties, much work has gone into an investigation of the properties of individual ideals on $P_\kappa\lambda$. Nevertheless, as pointed out by Kanamori in ***The higher infinite***, p. 351 (JSL LXI 334), for ideals on $P_\kappa\lambda$ "a full structure theory has yet to be worked out." The seven papers under review represent a selection of Abe's results in this area.

Suppose κ is λ-supercompact. Menas (*A combinatorial property of $p_\kappa\lambda$*, JSL LVI 1098) showed that if λ is a strong limit cardinal of cofinality $< \kappa$, then $P_\kappa\lambda$ bears two distinct isomorphic prime ideals extending the non-stationary ideal $\mathrm{NS}_{\kappa\lambda}$. In the first paper the author proves that the conclusion still holds if λ is inaccessible.

The remainder of the paper and the next two papers are concerned with Kanamori ideals. An ideal J on $P_\kappa\lambda$ is *Kanamori* if for every regressive $f : P_\kappa\lambda \to \lambda$, there exist $B \in J^*$ and $\gamma < \lambda$ such that $f(b) \leqq \gamma$ for every $b \in B$. (The corresponding property for ideals on κ was studied by Kanamori in *Weakly normal filters and irregular ultrafilters*, ***Transactions of the American Mathematical Society***, vol. 220 (1976), pp. 393–399, whence the name). J is *weakly normal* if given $A \in J^+$ and a regressive $f : P_\kappa\lambda \to \lambda$, there exist $B \in J^+ \cap P(A)$ and $\gamma < \lambda$ such that $f(b) \leqq \gamma$ for every $b \in B$. The two notions appear in the literature under various names. Abe himself varies in his terminology. By a "weakly normal ideal" he means what I call a Kanamori ideal in the second paper and what I call a weakly normal ideal in the fifth one. Note that the concept of weak normality is useless if $\mathrm{cf}(\lambda) < \kappa$, since then *every* ideal on $P_\kappa\lambda$ is weakly normal. Also, the two notions coincide if J is prime : J is Kanamori iff it is weakly normal. More generally, as shown by Abe in the third paper, an ideal J on $P_\kappa\lambda$ is Kanamori iff it is weakly normal and every disjoint family of sets in J^+ has size $< \mathrm{cf}(\lambda)$.

In the first paper only *prime* Kanamori ideals are considered. There the author makes the simple but crucial observation that every prime ideal on $P_\kappa\lambda$ has a prime Kanamori ideal below it in the RK-ordering. Thus there always exists a prime Kanamori ideal on $P_\kappa\lambda$ in case κ is λ-compact. It is also shown that if κ is a measurable limit of supercompact cardinals and λ is regular, then there is a prime Kanamori ideal J on $P_\kappa\lambda$ that is RK-minimal but does not extend $\mathrm{NS}_{\kappa\lambda}$ (in fact $\{a \in P_\kappa\lambda : a \cap \kappa \in \kappa\} \in J$).

The second paper is concerned with existence of Kanamori ideals. Let $K_{\kappa\lambda}$ assert the existence of a Kanamori ideal on $P_\kappa\lambda$. Abe's first observation is that $K_{\kappa\lambda}$ implies that κ is a limit cardinal. He goes on to establish that if $K_{\kappa\lambda}$ holds and either $2^{<\mathrm{cf}(\lambda)} < \kappa$ or $\mathrm{cf}(\lambda) = \kappa$, κ is inaccessible, and there is no Suslin κ-tree, then κ is λ-compact. Finally, he shows that these results are sharp by establishing the consistency of the following three statements (relative to the existence of a supercompact cardinal): (A) $\kappa = \mathfrak{c}$ and $K_{\kappa\lambda}$ holds for every $\lambda > \kappa$ of uncountable cofinality. (B) κ is inaccessible, there is a Suslin κ-tree and $K_{\kappa\lambda}$ holds for every $\lambda > \kappa$ of cofinality $\geqq \kappa$. (C) κ is weakly compact but not measurable and $K_{\kappa\kappa^+}$ holds. An intriguing recent result of D. Burke (*A note on a question of Abe*, ***Fundamenta mathematicae***, vol. 163 (2000), pp. 95–98) is that $K_{\kappa\lambda}$ holds if λ is regular *and less than some measurable cardinal* and there is an ideal J such that every disjoint family of sets in J^+ has size $< \lambda$.

Solovay (*Strongly compact cardinals and the GCH*, ***Proceedings of the Tarski Symposium***, American Mathematical Society, 1974, pp. 365–372) proved that if κ is λ-compact and λ is regular, then $\lambda^{<\kappa} = \lambda$. In the third paper, Abe generalizes this result by showing that if $K_{\kappa\lambda}$ holds and λ is regular (or $\mathrm{cf}(\lambda) = \kappa$), then $\lambda^{<\kappa} = \lambda \cdot 2^{<\kappa}$. He also shows that if $K_{\kappa\lambda}$ holds and $\mathrm{cf}(\lambda) < \kappa$, then $(\lambda^+)^{<\kappa} = \lambda^+ \cdot 2^{<\kappa}$.

The fourth and fifth papers are mostly concerned with $[\lambda]^{<\kappa}$-normal (also called strongly normal) ideals. Given an infinite cardinal $\mu \leqq \kappa$, an ideal J on $P_\kappa\lambda$ is $[\lambda]^{<\mu}$*-normal* if for every $A \in J^+$ and every $f : A \to P_\mu\lambda$ that is regressive in the sense that $f(a) \in P_{|a\cap\mu|}(a)$ for all $a \in A$, there is $B \in J^+ \cap P(A)$ such that f is constant on B. It is easy to see that J is $[\lambda]^{<\omega}$-normal iff it is normal. Di Prisco and Marek (*Some aspects of the theory of large cardinals*, ***Mathematical logic and formal systems***, Marcel Dekker, 1985, pp. 87–139) observed that any prime normal ideal on $P_\kappa\lambda$ is $[\lambda]^{<\kappa}$-normal. Carr, Levinski, and Pelletier (*On the existence of strongly normal ideals over $P_\kappa\lambda$*, ***Archive for mathematical logic***, vol. 30 (1990), pp. 59–72) showed that if there is a $[\lambda]^{<\kappa}$-normal ideal on $P_\kappa\lambda$, then there is a smallest one (denoted by $\mathrm{NS}_{\kappa\lambda}^{[\lambda]^{<\kappa}}$).

Suppose κ is Mahlo. Then, as discovered by Abe, λ and $\lambda^{<\kappa}$ are in some sense closely related. In the fourth paper he proves that given any bijection j from $\lambda^{<\kappa}$ onto $P_\kappa\lambda$, $f^*(\mathrm{NS}_{\kappa\lambda}^{[\lambda]^{<\kappa}}) = \mathrm{NS}_{\kappa\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}}$, where $f : P_\kappa\lambda \to P_\kappa\lambda^{<\kappa}$ is defined by $f(a) = \{\alpha < \lambda^{<\kappa} : j(\alpha) \in P_{|a\cap\kappa|}(a)\}$. He also shows that for every $[\lambda]^{<\kappa}$-normal ideal J on $P_\kappa\lambda$, either $\mathrm{sat}(J) \leqq \lambda$ or $\mathrm{sat}(J) \geqq \lambda^{<\kappa}$. Moreover, $\mathrm{sat}(\mathrm{NS}_{\kappa\lambda}^{[\lambda]^{<\kappa}}) > \lambda^{<\kappa}$.

Suppose λ is regular. Solovay (op. cit.) proved that if J is a prime normal ideal on $P_\kappa\lambda$, then the function $a \mapsto \bigcup a$ is one-to-one on some $A \in J^*$. In the fifth paper Abe generalizes this result by showing that the conclusion still holds if J is a λ-saturated $[\lambda]^{<\omega_1}$-normal ideal on $P_\kappa\lambda$. Moreover, he shows that any such J is RK-minimal.

The sixth paper deals with partition properties. Given an ideal J on $P_\kappa\lambda$ and $A \in J^+$, $A \to (J^+)^2$ means that for every $F : A \times A \to 2$, there are $B \in J^+ \cap P(A)$ and $i < 2$ such that $F(a, b) = i$ for all $a, b \in B$ with $a \subset b$. $J^+ \to (J^+)^2$ means that $A \to (J^+)^2$ for all $A \in J^+$. The author shows that if $I_{\kappa\lambda}^+ \to (I_{\kappa\lambda}^+)^2$ (or $I_{\kappa\lambda}$ is $(\lambda, 2)$-distributive), where $I_{\kappa\lambda}$ denotes the ideal of non-cofinal subsets of $P_\kappa\lambda$, then $\lambda^{<\kappa} < 2^\lambda$, which answers a question of Johnson (*Some partition relations for ideals on $P_\kappa\lambda$*, ***Acta mathematica Hungarica***, vol. 56 (1990), pp. 269–282). (Let me note in passing that the proof of Lemma 2.1 shows only that $I_{\kappa\lambda}|X = \mathrm{NS}_{\kappa\lambda}|X$ for *some* $X \in \mathrm{NS}_{\kappa\lambda}^+$.) Furthermore, he shows that if $I_{\kappa\lambda}^+ \to (I_{\kappa\lambda}^+)^2$ (or $I_{\kappa\lambda}$ is $(\lambda, 2)$-distributive) and $\lambda^{<\lambda} = \lambda$, then $P_\kappa\mu$ bears a normal $(\mu, 2)$-distributive ideal for every cardinal μ with $\kappa \leqq \mu < \lambda$. (By a recent result of Shioya, *Partition properties of subsets of $\mathcal{P}_\kappa\lambda$*, ***Fundamenta mathematicae***, vol. 161 (1999), pp. 325–329, it is consistent—relative to the existence of a supercompact cardinal—that $I_{\kappa\kappa^+}^+ \to (I_{\kappa\kappa^+}^+)^2$.) Abe also improves a result of Carr (*$P_\kappa\lambda$ partition relations*, JSL LVI 1097) by showing that $P_\kappa\lambda \to ((\nabla I_{\kappa\lambda})^+)^2$ implies that κ is λ-ineffable.

In order to discuss the seventh paper, let me abandon the convention made above and adopt a new one. By ideal on $P_\kappa\lambda$, I now mean a proper and κ-complete (but not necessarily fine) ideal on $P_\kappa\lambda$. Abe and Shioya prove that if J is a prime (κ, λ)-regular ideal on $P_\kappa\lambda$, then $j_J(\kappa) \geqq 2^{\lambda^{<\kappa}}$, where j_J is the elementary embedding induced by J. On the other hand, starting from a model where κ is a supercompact cardinal with a huge cardinal above it, they construct a generic extension where κ remains supercompact and, setting $\lambda = \kappa^{+++}$, show there is a prime uniform ideal J on $P_\kappa\lambda$ such that $j_J(\kappa) < 2^{\lambda^{<\kappa}}$ (in fact $j_J(\kappa) < \lambda^+$).

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