The Bulletin of Symbolic Logic
Volume 8, Number 2, June 2002

A GENERAL NOTION OF REALIZABILITY

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Abstract. We present a general notion of realizability encompassing both standard Kleene style realizability over partial combinatory algebras and Kleene style realizability over more general structures, including all partial cartesian closed categories. We show how the general notion of realizability can be used to get models of dependent predicate logic, thus obtaining as a corollary (the known result) that the category **Equ** of equilogical spaces models dependent predicate logic. Moreover, we characterize when the general notion of realizability gives rise to a topos, *i.e.*, a model of impredicative intuitionistic higher-order logic.

§1. Introduction. Work on realizability began around 1940 with the seminal paper of Kleene [14]. As explained by Kleene himself in his 1973 retrospective survey [15], Kleene developed his numerical realizability interpretation because he wished to make precise the intuitive connection between intuitionistic mathematics and recursive function theory, with both theories stressing the importance of extracting information effectively. In particular, he started by analyzing closed formulas of the form $\forall x.\exists y.\varphi(x, y)$ and conjectured that, if such a formula is true in intuitionistic number theory, then there must be a recursive choice function picking out a y as a function of x such that $\varphi(x, y)$ holds.

Kleene's realizability interpretation of intuitionistic number theory was extended to intuitionistic many-sorted higher-order logic by Hyland in the late 1970's [11]. In particular, Hyland showed how to use numerical realizability to construct the so-called *effective topos*[1] in which the standard truth-definition based on the natural numbers object is equivalent to Kleene's number realizability. We shall be concerned with similar constructions in the sequel, so let us outline how the construction of the effective topos goes. More information concerning the history and developments of realizability can be found in the two recent complementary excellent surveys [28, 29].

Received November 2, 2000; revised January 18, 2002.

A conference version of this paper was presented at the LICS 2000 conference [3].

*The work reported here was carried out while the author was a PhD-student at Carnegie Mellon University, Pittsburgh, USA.

[1]A topos is a certain kind of category with exactly the right properties to be a model of impredicative intuitionistic higher-order logic, see, *e.g.*, [16].

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1079-8986/02/0802-0003/$2.70

One first constructs a so-called *tripos*, call it P, which is a model of intuitionistic many-sorted higher-order logic in which the types are interpreted as ordinary sets, and in which a predicate $\varphi(x)$ with a free variable x†of type X is a set-theoretic function from X to $P\,\mathbb{N}$, the powerset of the natural numbers (here and in the sequel we often blur the distinction between syntax and semantics). In this model, the entailment $\varphi \vdash \psi$ of predicate ψ from ϕ over X holds iff there exists a number $n \in \mathbb{N}$ such that, for all $x \in X$ and for all $m \in \varphi(x)$, $n \cdot m$ is defined and $n \cdot m \in \psi(x)$, where $n \cdot m$ means "run the n'th Turing machine on the number m," the notion of application also originally used by Kleene.

A topos is a certain kind of category which models impredicative intuitionistic many-sorted higher-order logic in a standard way, namely by modelling the types as objects in the category and the terms as morphisms. A predicate $\varphi(x)$ with a free variable x of type X, corresponding to an object X, is modelled as a subobject of X (a subobject is just the categorical notion corresponding to a subset) or, equivalently, as a morphism from X into the object Ω of truth-values.

The effective topos is constructed from the tripos P in the following way. As objects in the category one takes partial equivalence relations in the logic of P, *i.e.*, a set X together with a predicate $\approx$ on $X \times X$ which is symmetric and transitive in the logic of P. To be, *e.g.*, symmetric *in the logic of* P means that the entailment $(x \approx x') \vdash (x' \approx x)$ holds in P. As morphisms in the category one takes (equivalence classes of) functional relations, again in the logic of P, between partial equivalence relations. One can then prove that the category obtained is a topos and that it really is a generalization of Kleene's number realizability, as mentioned above. In [11] Hyland also investigated full subcategories of the effective topos, in particular, the full subcategory of so-called assemblies and modest sets, which is often the most interesting for applications.

We now recall a few facts about the above construction and generalizations thereof.[2] The effective topos is referred to as a *realizability* topos, because it is a generalization of a known notion of realizability. Many other known notions of realizability can be generalized in a similar way. For example, one can obtain a topos generalizing Kleene's function realizability by substituting recursive functions for numbers and defining application $\cdot$ in the same way as Kleene did it in his function realizability. Both numbers (with the notion of application defined above) and recursive functions (with Kleene's notion of application) form examples of what is called a *partial combinatory algebra*. A **partial combinatory algebra** (PCA) consists of a set A together with a partial application function $\cdot : A \times A \rightharpoonup A$ and two distinct elements $\mathbf{K}, \mathbf{S} \in A$

[2]There are other (less logical, more categorical) constructions of the effective topos; for these one may make considerations similar to the ones we make here, see [3, 2].

satisfying, for all x, y, and z in A,

$$\mathbf{K} \cdot x \downarrow, \quad \mathbf{S} \cdot x \downarrow, \quad (\mathbf{S} \cdot x) \cdot y \downarrow,$$

and

$$(\mathbf{K} \cdot x) \cdot y \simeq x, \quad ((\mathbf{S} \cdot x) \cdot y) \cdot z \simeq (x \cdot z) \cdot (y \cdot z),$$

where $P \downarrow$ means that P is defined and where **Kleene equality** $P \simeq Q$ means that P is defined if and only if Q is defined, and in that case they are equal. One often simply writes xy for $x \cdot y$. Recall that a PCA A satisfies combinatory completeness: for every term M and any variable x, there is a term $\lambda x.\, M$ such that for any term N, $(\lambda x.\, M)N \simeq M[N/x]$. One also has a pairing operation $\langle x, y\rangle$ and projections π and π' satisfying that $\langle x, y\rangle \downarrow$; $\pi\langle x, y\rangle = x$; and $\pi'\langle x, y\rangle = y$.

For the PCA over the natural numbers, the **K** and **S** combinators are obtained using the S-M-N theorem. Other examples of PCA's (besides the two we have already mentioned) may be obtained from models of the untyped lambda calculus. Indeed, one can think of a partial combinatory algebra as a model of untyped computation.

The notion of tripos is due to Hyland, Johnstone and Pitts [12] who showed that given *any* partial combinatory algebra, one may construct a realizability tripos over it. They also proved that any tripos gives rise to a topos in the way outlined above and that certain relationships between given triposes lift to relationships between the induced toposes. In other words, triposes are used to analyze realizability toposes in much the same way as so-called sites are used to analyze Grothendieck toposes — a Grothendieck topos is a certain kind of topos which satisfies some extra conditions, *e.g.*, that it models infinite disjunctions. There is a precise definition of what a Grothendieck topos is (see, *e.g.*, [20]); any Grothendieck topos can be constructed from a site and Grothendieck toposes can be usefully analyzed by considering the corresponding sites. The situation is not so clear cut for realizability toposes: there is no precise definition of what a realizability topos is; certainly, one refers to toposes generated from PCA's in the way outlined above as realizability toposes, but there is no precise definition of when a given topos is a realizability topos. Finding a good definition would be useful, because it would give a better understanding of what characterizes realizability models of intuitionistic higher-order logic. The answer is not simply that a realizability topos is a topos constructed from a tripos, since also many Grothendieck toposes may be constructed from triposes. Thus there has been an interest in obtaining a better understanding of the construction of the realizability toposes obtained from partial combinatory algebras. In particular, one would like to understand *why* one obtains a topos in the first place, in particular, where does the impredicativity come from? This is one of the reasons for the recent interest in understanding how models based on realizability over partial combinatory algebras may be generalized to models based on realizability over more general structures [1, 4, 2, 19, 18, 17, 9]. Another reason is that it has been found that recently suggested categories, which, in a sense, only lack impredicativity to be toposes, can be usefully understood and analyzed as realizability categories. For instance, in [4, 1] it is shown how the recent category **Equ** of equilogical spaces, suggested by Dana Scott, can be thought of as a category of modest sets over the category of algebraic lattices and, proceeding by analogy to the category of modest sets constructed over a partial combinatory algebra, it is shown that **Equ** models dependent type theory. The category **Equ** is a cartesian closed extension of the category of topological T_0-spaces, so has finite products and well-behaved function spaces, and is of interest not only from a purely topological and logical perspective but also from a general computer science perspective since it contains many subcategories of domains as used in denotational semantics. In this paper we suggest a common framework which covers both Kleene style realizability over partial combinatory algebras and Kleene style realizability over more general "typed" structures, including the category **ALat** of algebraic lattices used for **Equ**. Moreover, we give a precise explanation of why one gets a topos when considering realizability over untyped models of computation (partial combinatory algebras) but not when considering models over typed models of computation.

At the core of standard realizability models over partial combinatory algebras is the basic Kleene realizability interpretation of implicational logic. Our general notion of realizability is based on the observation that for this interpretation of implicational logic, one only needs a few of all the definability properties a partial combinatory algebra provides. Indeed, intuitively speaking, we just need some kind of pairing operation for conjunction and some kind of functional abstraction for implication. Therefore, our general notion of realizability is based on the notion of a *weakly closed partial cartesian* (WCPC) category $\mathbb{C}$, equipped with a functor $U : \mathbb{C} \to \mathbf{Ptl}$ into the category of sets and partial functions (see the following sections for precise definitions). We may think of a WCPC-category $\mathbb{C}$ as a general universe of realizers as follows. The objects are thought of as sets of realizers and the morphisms are thought of as realizable partial functions mapping one set of realizers into another; partial because realizers may be only partially defined. A WCPC-category is equipped with a kind of pairing operation (hence the word "cartesian") and with a kind of weak functional abstraction (hence the phrase "weakly closed"). Since a WCPC-category may have many different objects, one may also think of a WCPC-category as a *typed* universe of realizers. Any cartesian closed category is a WCPC-category and any partial combinatory algebra gives rise to a WCPC-category in a natural way.

We show in particular that (1) any WCPC-category $\mathbb{C}$ gives rise to a pre-tripos (a model of many-sorted predicate logic in which sorts are interpreted by ordinary sets); (2) any such realizability pre-tripos gives rise to categories

Mod($\mathbb{C}$) and **Asm**($\mathbb{C}$) of modest sets and assemblies over $\mathbb{C}$ which model dependent predicate logic; (3) a WCPC-category gives rise to a realizability topos iff it has a universal object, that is, an object of which all the other objects are retracts; and (4) a PCA A gives rise to a WCPC-category $\mathbb{C}(A)$ and realizability in the sense of the WCPC-category $\mathbb{C}(A)$ is indeed equivalent to realizability in the sense of the PCA A.

As a corollary of (2), we get that **Equ** $\simeq$ **Mod**(**ALat**) models dependent predicate logic as shown explicitly in [1]. Intuitively, (3) this tells us that we only get an impredicative model (a topos) if we start out with an impredicative (untyped) universe of realizers. In particular, since **ALat** does not have a universal object, we conclude that realizability over **ALat** does not give rise to a topos, and thus **Equ** does not embed into a realizability topos constructed over **ALat** (this was known before; the point is that our result *explains* why this particular example does not produce a topos). Finally, (4) tells us that our framework is a proper generalization of standard realizability over PCAs, and hence it improves earlier published work such as [4].

Throughout the paper we make use of categorical logic and type theory: when we claim that we get a model of a certain type theory and logic, then what we actually prove is that we get a category with certain properties which, by known results in categorical logic and type theory, tells us that the category is indeed a model. For good introductions to categorical type theory and logic, see, *e.g.*, [16, 13, 27].

Independently of the work reported here, John Longley has recently suggested another general framework encompassing both typed and untyped realizability [18]. Longley's definition of a *partial combinatory type structure* (PCTS) is essentially a typed version of a PCA, with **K** and **S** combinators for all appropriate types. For any such PCTS, Longley then defines a category of assemblies and shows that it is regular and locally cartesian closed. We leave a detailed analysis of the relationship between Longley's notion of PCTS and our notion of WCPC-category and the induced notions of categories of assemblies for future work.

Peter Lietz and Thomas Streicher [17] have, in the context of Longley's PCTS's, shown some results closely related to our Theorems 3.6 and 3.8 characterizing when the tripos-to-topos construction applied to the realizability tripos over a WCPC-category yields a topos. In particular, Lietz and Streicher show that a PCTS has a universal type iff it gives rise to a topos and iff it is equivalent (in a suitable sense) to a PCA (obtained from the universal object). This result in fact indicates that our framework of WCPC-categories may be slightly more general than Longley's PCTS's because one can define a WCPC-category which gives rise to a topos, which is *not* a standard realizability topos over a PCA [2, Remark 6.0.4].

1.1. Outline. In Section 2 we define the notion of a WCPC-category and show how PCA's give rise to WCPC-categories. Then in Section 3 we define the notion of a pretripos and show that any WCPC-category gives rise to what we call a realizability pretripos. We further show that the pretripos resulting from the WCPC-category induced by a PCA is equivalent to the standard realizability tripos over the PCA, and we give a general explanation of this fact. Further, we show that a realizability pretripos gives rise to a topos iff the WCPC-category has a universal object. In Section 4 we go on to define categories of assemblies over realizability pretriposes and we show that the category of assemblies over a realizability pretripos models dependent predicate logic. Finally, in Section 5 we discuss how our general notion of realizability is related to other recently suggested frameworks and give some concluding remarks.

Acknowledgements. The work reported here forms part of the author's PhD-thesis [2], written under the guidance of Prof. Dana Scott. We refer the reader to [3, 2] for more details than we can include here. I am also grateful for useful discussions with Steven Awodey, Andrej Bauer, Peter Lietz, John Longley, Pino Rosolini, and Thomas Streicher. Finally, I would like to thank the referees for their useful comments.

§2. WCPC-categories. In this section we define the notion of a *weakly closed partial cartesian category* (WCPC-category). We shall show how such a category can be seen as a general universe of realizers. In particular, we show that any PCA gives rise to a WCPC-category.

Our notion of a WCPC-category is just a weak version of a category of partial maps on a cartesian closed category. We thus begin by recalling the definition of a partial cartesian category from [23]. A cartesian category is a category with finite products and the wording "partial cartesian" is a shortening of "partial maps on a cartesian category." There is an excellent overview of categories of partial maps in [23] to which we refer the reader for much more information on categories of partial maps and their history than we can include here. See also [25]. The following definitions are from [23, Page 101].

Definition 2.1. A p**-category** is a category $\mathbb{C}$ endowed with a bifunctor $\times\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ which is called **product**, a natural transformation $\Delta\colon (-)\to(-\times-)$ which is called the **diagonal** and two families of natural transformations $\{p_{-,Y}\colon (-\times Y)\to(-)\mid Y\in\mathbb{C}\}$ and $\{q_{X,-}\colon (X\times -)\to(-)\mid X\in\mathbb{C}\}$ which are called **projections**, satisfying the identities

$$
\begin{array}{ll}
p_{X,X}\Delta_X = id_X = q_{X,X}\Delta_X & (p_{X,Y}\times q_{X,Y})\Delta_{X\times Y} = id_{X\times Y}\\
p_{X,Y}(id_X\times p_{Y,Z}) = p_{X,Y\times Z} & p_{X,Z}(id_X\times q_{Y,Z}) = p_{X,Y\times Z}\\
q_{X,Z}(p_{X,Y}\times id_Z) = q_{X\times Y,Z} & q_{Y,Z}(q_{X,Y}\times id_Z) = q_{X\times Y,Z}.
\end{array}
$$

Finally we require that the associativity and commutativity isomorphisms α and τ defined as below by

$$\alpha_{X,Y,Z} = ((id_X \times p_{Y,Z}) \times q_{Y,Z} q_{X,Y\times Z})\Delta_{X\times(Y\times Z)}\colon X \times (Y \times Z) \to (X \times Y) \times Z$$

and

$$\tau_{X,Y} = (q_{X,Y} \times p_{X,Y})\Delta_{X\times Y}\colon X \times Y \to Y \times X$$

are natural in all variables (the components need not be so).

A p-functor $F\colon \mathbb{C} \to \mathbb{D}$ between p-categories is a functor preserving products up to a natural isomorphism which is coherent with projections and diagonals.

Given a map $f: X \to Y$ in the p-category $\mathbb{C}$, the **domain** dom $f: X \to Y$ of f is the composite map $p_{X,Y}(id_X \times f)\Delta_X\colon X \to X$. A map $f: X \to Y$ in a p-category $\mathbb{C}$ is **total** if dom $f = id$. One shows that total maps form a subcategory $\mathbb{C}_t$. Note that a category with binary products is trivially a p-category whose maps are all total.

EXAMPLE 2.2. Let **Ptl** denote the category of sets and partial functions. It is a p-category with total category the usual category **Set** of sets and total functions.

For all objects X and Y in a p-category $\mathbb{C}$, the maps id_X, $p_{X,Y}$, $q_{X,Y}$, and Δ_X are total. If morphisms $f: X \to Y$ and $g\colon Y \to Z$ are both total, then their composite $gf: X \to Z$ is total. The subcategory $\mathbb{C}_t$ of total maps has binary cartesian products. A p-functor maps total maps to total maps.

A p-category $\mathbb{C}$ is said to be a **partial cartesian category** if the subcategory $\mathbb{C}_t$ of total maps has a terminal object (and so has finite products). The following definition is a weak version of the standard notion of a closed partial cartesian category [23].

DEFINITION 2.3. Let $\mathbb{C}$ be a partial cartesian category. Then we say that $\mathbb{C}$ is **weakly closed** if, for all X and Z in $\mathbb{C}$, there exists an object $[X \rightharpoonup Z]$ and a morphism $e\colon [X \rightharpoonup Z] \times X \to Z$ in $\mathbb{C}$ such that, for all objects Y, all morphisms $f\colon Y \times X \to Z$ in $\mathbb{C}$, there exists a (not necessarily unique) map $f'\colon Y \to [X \rightharpoonup Z]$ in $\mathbb{C}_t$ (note that f' is total!), such that the following diagram commutes in $\mathbb{C}$:

$$\begin{tikzcd}
{[X \rightharpoonup Z] \times X} \arrow[r, "e"] & Z \\
Y \times X \arrow[u, "f' \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We refer to $([X \rightharpoonup Z], e)$ as the **weak partial exponential of X and Z.**

For simplicity we refer to a weakly closed partial cartesian category as a WCPC-category. In case f' in the above definition exists *uniquely*, one says that the category is closed and in the simple case of a category with finite products (with all maps total), the notion of being closed is the standard one of cartesian closure. In other words, any cartesian closed category is a WCPC-category and thus, since the category **ALat** of algebraic lattices and Scott continuous functions is cartesian closed [8, 10] it is a WCPC-category.

EXAMPLE 2.4. The category **Ptl** of sets and partial functions is a WCPC-category. In fact, it is not only weakly closed but closed. Indeed, the product $X \times Y$ of sets X and Y is just the standard set-theoretic cartesian product and the exponential object $[X \rightharpoonup Z]$ is the set of partial functions from X to Z.

We may think of a WCPC-category $\mathbb{C}$ as a general universe of realizers as follows. The category $\mathbb{C}$ is a p-category because realizers may be only partially defined. Intuitively, $[X \rightharpoonup Z]$ is a set of realizers of functions from X to Z. There may be more than one realizer for each function from X to Z, hence f' is only required to exist, not to be unique.

Next, we will show that any partial combinatory algebra gives rise to a WCPC-category, which is *not* necessarily closed. Let A be a PCA. By an **A-definable (total) function** we mean a function f from A to A for which there exists an element $a \in A$ such that, for all $b \in A$, $a \cdot b \downarrow$ and $a \cdot b = f(b)$. The A-definable functions form a monoid under composition; we refer to this monoid as the **monoid of A-definable functions**. By an **A-definable partial function** we mean a partial function f from A to A for which there exists an element $a \in A$ such that, for all $b \in A$, we have that $a \cdot b \simeq f(b)$.

DEFINITION 2.5. Let A be a PCA.

1. We define the category $\mathbb{C}(A)_t$ to be the idempotent splitting of the monoid of A-definable total functions. That is, objects are total A-definable idempotents and a morphism $f : X \to Y$ is an A-definable total function such that $YfX = f$. The identity on X is X itself and composition is composition of total functions.
2. We define the **category** $\mathbb{C}(A)$ **induced by** A to be the category with objects total A-definable idempotents (*i.e.*, same objects as $\mathbb{C}(A)_t$) and with morphisms $f : X \to Y$ A-definable partial functions f satisfying that $YfX = f$ (as partial functions). The identity on X is X and composition is composition of partial functions.

This definition is related to and inspired by the work of Dana Scott [26] and the treatment of C-monoids in [16].

PROPOSITION 2.6. *Let A be a PCA. Then the category $\mathbb{C}(A)$ induced by A is a WCPC-category and its subcategory of total maps is equivalent to $\mathbb{C}(A)_t$.*

See [2] for the proof, which involves some of the usual calculations with idempotents as in [26, 16]. We will often refer to the category $\mathbb{C}(A)$ induced by a PCA A as **the WCPC-category induced by** A, thus implicitly referring to the above proposition.

For a PCA A, we define a p-functor $U\colon \mathbb{C}(A) \to \mathbf{Ptl}$ as follows. On an object $X \in \mathbb{C}(A)$, we set

$$U(X) = \{X(a) \mid a \in A\},$$

and for a morphism $f : X \to Y$ in $\mathbb{C}(A)$, we set $U(f)$ to be f (appropriately restricted and corestricted). Note that U indeed is a p-functor; in particular U maps $\mathbb{C}(A)_t$ into **Set** and U applied to the terminal object in $\mathbb{C}(A)_t$ is a terminal object in **Set**. Further note that U is faithful, so it really gives us a representation of $\mathbb{C}(A)$ as a category of partial maps. This will be useful later on, see Theorem 3.6.

§3. Pretriposes.

Definition 3.1. A **Set-indexed preorder** is a functor $P\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ whose fibres $P(I)$, for $I \in \mathbf{Set}$, are all preorders. For such an indexed preorder P, if $u\colon I \to J$ in **Set**, then we often write u^* for the monotone function $P(u) : P(J) \to P(I)$ and refer to it as the **reindexing functor**. A **Set**-indexed preorder P is called a **pretripos** if

1. P is fibred cartesian closed, *i.e.*, each fibre $P(I)$ is cartesian closed and the reindexing functors u^* preserves this structure (the fibred cartesian closed structure is for interpreting $\top$, $\wedge$, $\supset$);
2. each reindexing functor u^*, for $u\colon I \to J$ in **Set**, has left and right adjoints $\exists_u \dashv u^* \dashv \forall_u$, *i.e.*, for all $\varphi \in P(I)$, and $\psi \in P(J)$,

$$\exists_u(\varphi) \vdash_J \psi \quad \text{iff} \quad \varphi \vdash_I u^*(\psi)$$

and

$$\psi \vdash_J \forall_u(\varphi) \quad \text{iff} \quad u^*(\psi) \vdash_I \varphi.$$

Moreover, these adjoints are required to satisfy the Beck-Chevalley condition: for any pullback of the form

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "s"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

in **Set**, we have isomorphisms

$$s^*\forall_u \cong \forall_v r^* \qquad \text{and} \qquad \exists_v r^* \cong s^*\exists_u,$$

where $\vdash_I$ denotes the ordering in $P(I)$ and $\vdash_J$ denotes the ordering in $P(J)$.

If, in addition, P has fibred finite coproducts, *i.e.*, finite coproducts in each fibre and reindexing functors preserve these, (for $\bot$, $\vee$), then we say that the pretripos is a **pretripos with disjunction**.

Finally, a **tripos** (over **Set**) is a pretripos with disjunction, which has a weak generic object, *i.e.*, an object $\sigma \in P(\Sigma)$, for some $\Sigma \in \mathbf{Set}$, such that given any $\varphi \in P(I)$, there is a map $\{\varphi\}\colon I \to \Sigma$ in **Set** with $\{\varphi\}^*(\sigma) \cong \varphi$.

In a tripos, one may think of Σ as the set of truth values. Note that since Σ is an object of **Set** one may quantify over Σ and also over, *e.g.*, the set Σ^I of functions from I to Σ (predicates) — in this way a tripos models impredicative intuitionistic higher-order logic.

The name "pretripos" has been chosen to reflect the fact that such an indexed preorder is almost a tripos [12, 21]; the essential difference is that a pretripos with disjunction is not required to have a weak generic object, *i.e.*, a pretripos does not model impredicative higher-order logic, but just predicate logic over a higher-order language.

3.1. Realizability pretriposes. Let $\mathbb{C}$ be a WCPC-category and $U\colon \mathbb{C} \to$ **Ptl** a p-functor. We now show how to define a pretripos $P_{\mathbb{C}}$ from $\mathbb{C}$ and U. We refer to it as the **realizability pretripos over** $\mathbb{C}$, thus omitting explicit mentioning of U (which we think of as a forgetful functor). The idea is as for realizability triposes over partial combinatory algebras [12], the difference being that here the realizers will be drawn from $\mathbb{C}$ instead of from a PCA. By a realizer we shall mean a morphism f in $\mathbb{C}$.

We define $P_{\mathbb{C}}\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ as follows. For a set I, $P_{\mathbb{C}}(I)$ is the preorder with objects pairs of the form (A, φ), where A is an object of $\mathbb{C}$ and $\varphi\colon I \to P\,(UA)$ in **Set**, where $P\,(UA)$ is the ordinary powerset of UA. The less-than-or-equal relation is denoted $\vdash$ and the objects are preordered by decreeing that $(A, \varphi) \vdash (B, \psi)$ if and only if

$$\exists g \in \mathbb{C}(A, B).\ \forall i \in I.\ \forall a \in \varphi(i).\ U(g)(a)\downarrow \text{ and } U(g)(a) \in \psi(i).$$

We refer to the objects of $P_{\mathbb{C}}(I)$ as **predicates on** I or just as **predicates**. When the object A is clear from context we sometimes just write φ for (A, φ). Further, we sometimes write $(A, \varphi)(i)$ for $\varphi(i)$. We refer to the object A in (A, φ) as the **underlying object of realizers for** φ.

For a morphism $u\colon I \to J$, $P_{\mathbb{C}}(u)$ is essentially composition:

$$P_{\mathbb{C}}(u)(A, \varphi\colon J \to P\,(UA)) = (A, \varphi \circ u\colon I \to P\,(UA)).$$

Note that $P_{\mathbb{C}}(u)$ is indeed monotone, hence a well-defined functor. Moreover, $P_{\mathbb{C}}$ is clearly a functor.

The fibred cartesian closed structure is given as follows. In $P_{\mathbb{C}}(I)$ the terminal object is

$$\top = (1, i \mapsto U1),$$

where 1 is the terminal object in the category of total maps $\mathbb{C}_t$. For predicates (A, φ) and (B, ψ) over I, define

$$
\begin{aligned}
(A,\varphi) \wedge (B,\psi) &= (A \times B, i \mapsto \{(a,b) \mid a \in \varphi(i) \text{ and } b \in \psi(i)\}) \\
(A,\varphi) \supset (B,\psi) &= \;([A \rightharpoonup B], \\
&\qquad i \mapsto \{g \in U[A \rightharpoonup B] \mid \forall a \in \varphi(i).\ U(e)(g,a)\downarrow \text{ and} \\
&\qquad\qquad U(e)(g,a) \in \psi(i)\}),
\end{aligned}
$$

These definitions make sense because U preserves products, and they are easily shown to give binary products and exponentials in $P_{\mathbb{C}}(I)$. Quantifiers can also easily be defined, see [2].

Summarizing we have the following theorem.

THEOREM 3.2. *Let $\mathbb{C}$ be a WCPC-category. Then $P_{\mathbb{C}}$ is a pretripos.*

For a WCPC-category $\mathbb{C}$ and a morphism $U: \mathbb{C} \to \mathbf{Ptl}$ of WCPC-categories, we refer to the induced pretripos $P_{\mathbb{C}}$ as the **realizability pretripos over** $\mathbb{C}$.

We now present our two main example of realizability pretriposes.

EXAMPLE 3.3. For the category **ALat** of algebraic lattices, we define U: **ALat** $\to$ **Ptl** to be the composition of the forgetful functor from **ALat** to **Set** and the inclusion functor from **Set** into **Ptl**. When we refer to the realizability pretripos $P_{\mathbf{ALat}}$ over **ALat** it is always understood that we refer to this functor U: **ALat** $\to$ **Ptl** just defined.

EXAMPLE 3.4. By Proposition 2.6 we know that a PCA A generates a WCPC-category. When we refer to the realizability pretripos $P_{\mathbb{C}(A)}$ over the WCPC-category induced by A, the functor $U: \mathbb{C}(A) \to \mathbf{Ptl}$ is always assumed to be the functor U from Page 274. In this case, the realizability pretripos $P_{\mathbb{C}(A)}$ is equivalent to the standard realizability tripos over the partial combinatory algebra A, as defined in [12], *i.e.*, our definitions do indeed generalize the standard notions.

REMARK 3.5. One can define what it means for a p-category to have weak finite coproducts and show that if a WCPC-category has weak finite coproducts then the induced realizability pretripos has disjunction. We omit the details and just mention that both **ALat** and $\mathbb{C}(A)$ have weak finite coproducts [2].

3.2. Realizability pretriposes and universal objects. Example 3.4 is an instance of a more general phenomenon, which we now describe. Recall that a category $\mathbb{C}$ has a **universal object** V if all objects X in $\mathbb{C}$ are retracts of V. Observe that $\mathbb{C}(A)$ has a universal object, namely id_A. We can now show that, when U is faithful, the realizability pretripos over such a category $\mathbb{C}$ has a weak generic object just in case $\mathbb{C}$ has a universal object.

THEOREM 3.6. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a faithful p-functor. Then $\mathbb{C}$ has a universal object if and only if $P_{\mathbb{C}}$ has a weak generic object.*

See [2] for the proof. The following is now immediate from Theorem 3.6 and Remark 3.5: Let $\mathbb{C}$ be a WCPC-category with weak finite coproducts and suppose $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a faithful p-functor Then $\mathbb{C}$ has a universal object if and only if $P_{\mathbb{C}}$ is a tripos.

In fact, we can strengthen the above results to characterize exactly when the tripos-to-topos construction applied to $P_{\mathbb{C}}$ yields a topos. To do so we apply a result of Pitts who characterized (in 1982) when the tripos-to-topos construction applied to a pretripos with disjunction yields a topos [22]. For P a pretripos, we denote by $\mathbf{Set}[P]$ the category obtained by applying the tripos-to-topos construction to P. Pitts showed in particular that the tripos-to-topos construction applied to the realizabilty pretripos $P_{\mathbb{C}}$ (for $\mathbb{C}$ a WCPC-category) yields a topos iff the following comprehension axiom holds:[3]

AXIOM 3.7 (CA). *For all sets X there is a set PX and a predicate $In_X \in P_{\mathbb{C}}(X \times PX)$ such that, for any set I and $R \in P_{\mathbb{C}}(X \times I)$, the sentence*

$$\forall i\colon I.\ \exists s\colon PX.\ \forall x\colon X.\ (\mathrm{In}_X(x,s) \supset R(x,i)) \wedge (R(x,i) \supset \mathrm{In}_X(x,s))$$

is satisfied in $P_{\mathbb{C}}$.

Here we think of In_X as a membership predicate and of s as the set of all those elements of X for which R holds.

THEOREM 3.8. *Let $\mathbb{C}$ be a WCPC-category and suppose $U\colon \mathbb{C} \to \mathbf{Ptl}$ is a faithful p-functor. Then $\mathbb{C}$ has a universal object if and only if $P_{\mathbb{C}}$ satisfies the axiom (CA).*

PROOF. By Theorem 3.6, if $\mathbb{C}$ has a universal object, then $P_{\mathbb{C}}$ is a tripos and thus it also satisfies the axiom (CA).

For the other direction, suppose that $P_{\mathbb{C}}$ satisfies the axiom (CA). Let $X = 1$ and let Σ denote the object PX in (CA). Let $V \in \mathbb{C}$ be the underlying object of $\mathrm{In} = \mathrm{In}_1$, which exists because of (CA). We show that V is a universal object in $\mathbb{C}$. Let $A \in \mathbb{C}$ be arbitrary, let $I = UA$ and let $R(a) = \{a\}$. By (CA) the following sentence is valid in the realizability pretripos:

$$\forall a\colon UA.\ \exists s\colon \Sigma.\ \mathrm{In}(s) \supset\subset R(a).$$

Thus there is a realizer in

$$\bigcap_{a \in UA} \left(\bigcup_{s \in \Sigma} (\mathrm{In}(s) \supset R(a)) \wedge (R(a) \supset \mathrm{In}(s)) \right) \subseteq U([V \rightharpoonup A] \times [A \rightharpoonup V]).$$

[3]In fact, Pitts assumed that the pretripos has disjunction, but going through his proof one sees that disjunctions are not needed.

Using this realizer, which is a pair of morphisms between A and V, one shows easily that A is a retract of V. $\dashv$

COROLLARY 3.9. *For $\mathbb{C}$ be a WCPC-category and $U\colon \mathbb{C} \to$ **Ptl** a faithful p-functor, the tripos-to-topos construction applied to $P_{\mathbb{C}}$ yields a topos iff $\mathbb{C}$ has a universal object.*

A further corollary is that the tripos-to-topos construction applied to the realizability tripos induced by the category of algebraic lattices does not produce a topos, since clearly **ALat** does not have a universal object (for cardinality reasons).

§4. Assemblies. In this section we define a category of assemblies over a realizability pretripos. The definition can be given, and some of the properties proved, for more general structures (not only for realizability pretriposes), but, for concreteness, we choose here to only focus on assemblies over realizability pretriposes. See [2] for the more general results.

Let $\mathbb{C}$ be a WCPC-category and let $P_{\mathbb{C}}$ be the realizability pretripos over $\mathbb{C}$. Recall that for an object (A, E_X) of $P_{\mathbb{C}}(X)$ E_X is a function $E_X\colon X \to P(UA)$. We sometimes write such objects as (X, A, E_X).

DEFINITION 4.1. With notation as above, define $\mathbf{Asm}(\mathbb{C})$ to be the category with objects all objects (X, A, E_X) of $P_{\mathbb{C}}(X)$, for all $X \in \mathbf{Set}$, satisfying that, for all elements $x \in X$, $E_X(x) \neq \emptyset$. A morphism from (X, A, E_X) to (Y, B, E_Y) is a function $f\colon X \to Y$ in **Set** such that the sentence

$$\forall x\colon X.\ E_X(x) \supset E_Y(f(x))$$

is valid in $P_{\mathbb{C}}$.

Indeed the following two examples show how our setup specializes to the cases of assemblies over PCA's and over algebraic lattices.

EXAMPLE 4.2. Let A be a PCA. Recall from Example 3.4 that the induced realizability pretripos $P_{\mathbb{C}(A)}$ is equivalent to the standard realizability tripos over a A. Since the first-order structure is defined categorically, it is preserved by the equivalence functors. From the preceding it is easy to verify that the category $\mathbf{Asm}(\mathbb{C}(A))$ of assemblies over $\mathbb{C}(A)$ is equivalent to the usual category of assemblies $\mathbf{Asm}(A)$ over A.

EXAMPLE 4.3. Let $P_{\mathbf{ALat}}$ be the realizability pretripos over **ALat**. Then $\mathbf{Asm}(\mathbf{ALat})$ is equivalent to the category of assemblies over **ALat** defined in [1].

Recall that a category $\mathbb{C}$ is locally cartesian closed if every slice $\mathbb{C}/X$ category is cartesian closed. A locally cartesian closed category models dependent type theory. A category is regular if it models the fragment of many-sorted intuitionistic logic with $=$, $\top$, $\wedge$, and existential quantification $\exists$. If a category is not only regular but also locally cartesian closed it models, in addition, implication $\supset$ and universal quantification $\forall$. If the category in addition has finite coproducts then it also models $\perp$ and $\vee$, *i.e.*, then it models all of many-sorted intuitionistic predicate logic.

THEOREM 4.4. *Let $\mathbb{C}$ be a WCPC-category. Then* $\mathbf{Asm}(\mathbb{C})$ *is regular and locally cartesian closed. Moreover, if $\mathbb{C}$ has weak finite coproducts, then* $\mathbf{Asm}(\mathbb{C})$ *has stable and disjoint finite coproducts.*

For the proof (of a more general theorem), which makes use of the internal logic of pretriposes, see [2].

It follows that for $\mathbb{C}$ a WCPC-category with weak finite coproducts, $\mathbf{Asm}(\mathbb{C})$ models dependent predicate logic, *i.e.*, dependent type theory with predicate logic on top of it. See [13, Chapters 10-11] for a detailed description of dependent predicate logic. In fact, one can also model dependent subset types and dependent quotient types, see [2].

The category of modest sets $\mathbf{Mod}(\mathbb{C})$ is naturally defined to be the full subcategory of $\mathbf{Asm}(\mathbb{C})$ on those objects (X, A, E) for which $E(x) \cap E(x') \neq \emptyset \Longrightarrow x = x'$, for all $x, x' \in X$. This category also provides a model of dependent predicate logic; in the case where $\mathbb{C} = \mathbf{ALat}$, it specializes to the model for **Equ** [1]. The interpretation of dependent predicate logic in this category is written out explicitly in [2, Appendix A].

§5. Related work and concluding remarks. In our work with Carboni, Rosolini and Scott [4] we have developed a complementary approach to a general notion of realizability for type theory, based on the theory of exact categories and exact completions. The approach in [4, 7], is a generalization of "the exact-completion approach to realizability toposes" as found in [24, 5], whereas the approach in the present paper is a generalization of "the tripos-theoretic approach to realizability toposes" as found in [12, 21].

We briefly relate the approach in [4, 7] to the approach in this paper.

DEFINITION 5.1. Let $\mathbb{C}$ be a WCPC-category and $U : \mathbb{C} \to \mathbf{Ptl}$ a p-functor. The category $\mathcal{F}(\mathbb{C})$ is the category with objects triples (X, A, σ), where $X \in \mathbf{Set}$, $A \in \mathbb{C}$, and $\sigma : X \to U(A)$ in **Set**. Morphisms $f : (X, A, \sigma) \to (Y, B, \tau)$ are functions $f : X \to Y$ in **Set** such that there exists a $g : A \to B$ in $\mathbb{C}$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \arrow[d, "\tau"] \\ U(A) \arrow[r, "U(g)"'] & U(B) \end{tikzcd}$$

commutes in **Ptl** (note that only $U(g)$ may be partial).

Let A be a PCA and let $\mathbb{C}(A)$ be the WCPC-category induced by A. Then $\mathcal{F}(\mathbb{C}(A))$ is equivalent to the category of *partitioned assemblies* over A, as defined in [6, 24].

It is straightforward to show that, for $\mathbb{C}$ a WCPC-category and $U : \mathbb{C} \to \mathbf{Ptl}$ is a p-functor, $\mathcal{F}(\mathbb{C})$ has all finite limits. Recall that, for a left exact category $\mathbb{C}$, $(\mathbb{C})_{\mathrm{reg/lex}}$ denotes the regular completion of $\mathbb{C}$ and $(\mathbb{C})_{\mathrm{ex/lex}}$ denotes the exact completion of $\mathbb{C}$. Likewise, for a regular category $\mathbb{C}$, $(\mathbb{C})_{\mathrm{ex/reg}}$ denotes the exact completion of $\mathbb{C}$. The following is an easy generalization of [5, Lemma 6.1].

PROPOSITION 5.2. *The category* $\mathbf{Asm}(\mathbb{C})$ *is equivalent to* $(\mathcal{F}(\mathbb{C}))_{reg/lex}$.

It follows that, for $\mathbb{C}$ a WCPC-category, $U : \mathbb{C} \to \mathbf{Ptl}$ a p-functor, and $P_{\mathbb{C}}$ the induced realizability pretripos, the categories $\mathbf{Set}[P_{\mathbb{C}}]$, $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$, and $(\mathcal{F}(\mathbb{C}))_{\mathrm{ex/lex}}$ are all equivalent. Hence we have the following corollary:

COROLLARY 5.3. *If* $\mathbb{C}$ *is a WCPC-category and* $U : \mathbb{C} \to \mathbf{Ptl}$ *is a faithful p-functor, then* $(\mathcal{F}(\mathbb{C}))_{ex/lex}$ *is a topos iff* $\mathbb{C}$ *has a universal object.*

In a recent manuscript [9], de Marchi, Robinson and Rosolini have analyzed the $\mathcal{F}$-construction in more detail, one goal being to investigate exactly what conditions on $\mathbb{C}$ are forced for $\mathcal{F}(\mathbb{C})$ to be a topos (*i.e.*, they do not a priori assume that $\mathbb{C}$ is WCPC-category). What they find is that $\mathbb{C}$ *essentially* has to be a WCPC-category (there is a mild relaxation on the underlying p-category structure) with a universal object. They show that this means that $\mathbb{C}$ is essentially equivalent to $\mathbb{C}(A)$ for some PCA A (obtained from the universal object).

There are many interesting questions for future work. In particular, even though we now have a better understanding of *why* the construction of a realizability topos over a partial combinatory algebra actually results in a topos, we still do not have a good definition of what a realizability topos is, and thus we still lack a better understanding of what characterizes realizability models of impredicative intuitionistic higher-order logic.

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