The Bulletin of Symbolic Logic
Volume 8, Number 1, March 2002

GROUPS AND ALGEBRAS OF BINARY RELATIONS

STEVEN GIVANT AND HAJNAL ANDRÉKA

Abstract. In 1941, Tarski published an abstract, finitely axiomatized version of the theory of binary relations, called the theory of *relation algebras*. He asked whether every model of his abstract theory could be represented as a concrete algebra of binary relations. He and Jónsson obtained some initial, positive results for special classes of abstract relation algebras. But Lyndon showed, in 1950, that in general the answer to Tarski's question is negative. Monk proved later that the answer remains negative even if one adjoins finitely many new axioms to Tarski's system. In this paper we describe a far-reaching generalization of the positive results of Jónsson and Tarski, as well as of some later, related results of Maddux. We construct a class of concrete models of Tarski's axioms—called *coset relation algebras*—that are very close in spirit to algebras of binary relations, but are built using systems of groups and cosets instead of elements of a base set. The models include all algebras of binary relations, and many non-representable relation algebras as well. We prove that every atomic relation algebra satisfying a certain measurability condition—a condition generalizing the conditions imposed by Jónsson and Tarski—is essentially isomorphic to a coset relation algebra. The theorem raises the possibility of providing a positive solution to Tarski's problem by using coset relation algebras instead of the standard algebras of binary relations.

The first steps in developing a mathematical theory of binary relations were taken by Augustus De Morgan [4] in 1864, not long after George Boole's pioneering work [2] in developing an algebra of logic (or, taking a different point of view, a calculus of sets). A proper foundation of this theory was laid by Charles Sanders Peirce during the period 1872–1883 (see, in particular, [13]), and extensively developed by the German mathematician Ernst Schröder in the 649 page book [14] published in 1895.

The basic objects of study in this theory are binary relations between elements of some base set. The fundamental operations of the theory include the Boolean operations of forming the union $R \cup S$ of two relations R and S, and the complement $\sim R$ of a relation R with respect to a fixed universal (largest) relation E (often, but not always, the set of all ordered pairs of elements from the base set). There are also operations specific to binary relations: the operation of forming the *relational composition* of two relations

Received March 8, 2001; revised October 18, 2001.

This research was partially supported by Mills College and the Hungarian National Foundation for Scientific Research, Grants T30314 and T35192.

1079-8986/02/0801-0002/$3.70

R and S,

$$R \mid S = \{(x, y)\colon \exists z[(x, z) \in R \text{ and } (z, y) \in S]\}\,,$$

and the operation of forming the *inverse* (or *converse*) of a relation R,

$$R^{-1} = \{(y, x)\colon (x, y) \in R\}\,.$$

Among the distinguished relations are the identity relation *Id* on the base set. Using modern terminology, one could say that the early researchers were studying algebras

$$\mathfrak{A} = \langle A, \cup, \sim, |, ^{-1}, Id\rangle$$

of binary relations (where A is some collection of binary relations).

These early researchers were interested in relational notions that are expressible (as equations) in the calculus of relations. For instance, the notion of a relation R being a function can be expressed by the equation

$$R^{-1} \mid R \subseteq Id.$$

(We use the term "equation" because every Boolean inclusion $R \subseteq S$ can be written as an equation, for example, as the equation $R \cup S = S$.) The property of a non-empty relation being a point, that is, consisting of exactly one ordered pair, can be expressed (in the algebra of all binary relations on a base set) by the equation

$$(R^{-1} \mid E \mid R) \cup (R \mid E \mid R^{-1}) \subseteq Id$$

(where E denotes the universal relation on the base set). One can also express (equationally) the property of a non-empty relation consisting of exactly two ordered pairs, but not the property of consisting of exactly n ordered pairs for $n > 2$.

Another important interest of the founders of the calculus of relations was the variety of its laws. To establish the validity of a law, they argued set-theoretically, using the set-theoretical definitions of the operations involved. For example, to establish the Second Involution Law,

$$(R \mid S)^{-1} = S^{-1} \mid R^{-1},$$

they might have argued as follows:[1]

$$\begin{aligned}
(x, y) \in (R \mid S)^{-1} \quad &\text{if and only if} \quad (y, x) \in R \mid S \\
&\text{if and only if} \quad \exists z[(y, z) \in R \text{ and } (z, x) \in S] \\
&\text{if and only if} \quad \exists z[(z, y) \in R^{-1} \text{ and } (x, z) \in S^{-1}] \\
&\text{if and only if} \quad (x, y) \in (S^{-1} \mid R^{-1}).
\end{aligned}$$

After the work of Schröder, the theory went into an eclipse that seemed to signal its ultimate demise. However, a chance meeting in 1938 between

[1]We use a symbolism of more recent origin in this argument.

Alfred Tarski and Leopold Löwenheim changed this situation. Löwenheim gave Tarski a copy of an unpublished manuscript (later published in [8]) containing his thoughts on the possibility of developing all of mathematics in the framework of the calculus of relations.

Tarski seems to have been dissatisfied with the approach outlined by Löwenheim, and more generally with the approach to the calculus of relations as it occurred in the work of Schröder. He felt that, in deriving theorems, these authors made use of concepts and statements that were outside of the calculus. For example, to establish the validity of the Second Involution Law, they had to talk about individual elements of the universe of discourse, like x and y, that are nowhere mentioned in the statement of the theorem. In the 1941 article *On the Calculus of Relations* (see [15]), Tarski proposed a reformulation of the calculus as a theory of abstract algebras

$$\langle A, +, -, ;, {}^{\smile}, 1' \rangle$$

called *relation algebras*, just as the calculus of classes had been recast as an abstract theory of Boolean algebras. He proposed a finite set of axioms for his theory that eventually took the following equational form (see [3]): (i) (Boolean Laws) $\langle A, +, - \rangle$ is a Boolean algebra, (ii) (Associative Law) the operation ; of relative multiplication is associative, (iii) (Distributive Laws) the operations ; and ${}^{\smile}$ (conversion) are distributive over Boolean addition, (iv) (Identity Law) $a;1' = a$, (v) (First Involution Law) $a^{\smile\smile} = a$, (vi) (Second Involution Law) $(a;b)^{\smile} = b^{\smile};a^{\smile}$, (vii) (Cycle Law) $[a^{\smile};-(a;b)] + (-b) = -b$ (where a and b range over the elements of the universe A).

Of course, every algebra of binary relations, or *set relation algebra*, is a relation algebra in the sense that it is a model of Tarski's axioms. The set relation algebras play the role of the "standard" or "concrete" models of the theory.

Tarski wrote (in [15], pages 87–88) that he was practically sure that, on the basis of these axioms, he could derive all of the hundreds of laws to be found in Schröder's book. Despite this assurance, he was unable to prove that his axiom system really possessed the fundamental property of completeness.

TARSKI'S COMPLETENESS PROBLEM. *Is the set of axioms* (i)–(vii) *complete in the sense that every law true of all algebras of binary relations is derivable from it*?

A second related problem also occurred to Tarski: is the analogue of Stone's representation theorem for Boolean algebras true for abstract relation algebras?

TARSKI'S REPRESENTATION PROBLEM. *Is every abstract relation algebra representable as* (*i.e.*, *isomorphic to*) *an algebra of binary relations*?

Tarski did achieve a spectacular success with his theory. Around 1942, he showed that any reasonable system of set theory or number theory—and hence all of classical mathematics—can be formalized in the theory of relation algebras (provided we adjoin a distinguished constant intended to denote an abstract version of the membership relation "$\in$"). In fact, even the variables of the theory are unnecessary. All mathematical assertions can be formulated, and all mathematical theorems can be proved, in the finitary equational formalism of relation algebras, without any variables, sentential connectives, or quantifiers, and with the high school rule of replacement of equals by equals as the only rule of inference. Using this result, Tarski even constructed a fragment of the two-valued sentential calculus that is sufficient for the development of all of classical mathematics. (See [19] for details.)

In 1950, Lyndon [9] showed that both of Tarski's problems have negative solutions. He constructed a finite relation algebra in which some true equation (that is, an equation true in all algebras of binary relations) fails to hold. Actually, Lyndon claimed to prove more, namely that the class RRA of representable relation algebras is not first-order axiomatizable at all. But there was a mistake in his argument, for Tarski later proved, using his preservation theorem for universal theories (see [16] and [17]), that the class RRA is in fact axiomatizable by a set of equations (see [18]). In other words, RRA is a variety. From this theorem it follows (by Birkhoff's completeness theorem for equational theories—see [1]) that Tarski's completeness problem and representation problem are actually equivalent.

One might suspect that Lyndon's negative results are due to a flawed selection of axioms by Tarski. There could exist another, more powerful, finite set of axioms for which the completeness and representation problems have positive solutions. However, Monk [11] showed in 1964 that this is not the case: the theory of true equations is not finitely axiomatizable, or, expressed slightly differently, the class of algebras RRA is not finitely axiomatizable.

In the 1940s, before Lyndon obtained his negative results, Jónsson and Tarski seem to have been working towards a positive solution of Tarski's problems, and they obtained a number of important results. For example, they showed that every relation algebra can be embedded into (i.e., is isomorphic to a subalgebra of) a complete, atomic relation algebra, in much the same way that every Boolean algebra can be embedded into a complete, atomic Boolean algebra. (For Boolean algebras—but not for relation algebras—this embedding theorem is equivalent to the well-known assertion that every Boolean algebra is representable.) Generalizing their results, they developed a general theory of Boolean algebras with operators that today has found many applications in mathematics and computer science. (See [6].)

Jónsson and Tarski also obtained representation theorems for special classes of relation algebras (see [5] and [7]). For instance, they showed that every atomic relation algebra in which the atoms are functional is representable. (A element x is *functional* if it satisfies the equation $x^{\smile} ; x \leq 1'$.) They also proved that every atomic relation algebra in which the atoms are points is representable.[2] (A non-zero element x is a *point* if it satisfies the equation

$$(x^{\smile} ; 1 ; x) + (x ; 1 ; x^{\smile}) \leq 1',$$

where 1 denotes the Boolean unit.)

Maddux [10] extended this last theorem. He proved that any relation algebra (atomic or not) in which just the identity element (not the entire Boolean unit) is the sum of points is representable. Generalizing this theorem still further, he proved that "pair-dense" relation algebras—that is, algebras in which the identity element is the sum of points and "pairs"—must be representable.[3]

Below, we shall show how the representation theorems of Jónsson-Tarski (as formulated above) and of Maddux (in the key case when the algebras are atomic) are all special cases of a much more general representation theorem in which the points of an algebra of relations are replaced by groups. Already in his 1941 article, Tarski remarked that the theory of relation algebras seemed to be a kind of union of the theories of Boolean algebras and of

[2]In fact, Jónsson and Tarski characterized the complete, atomic relation algebras in which the atoms are points: they are the isomorphic copies of algebras of binary relations in which the universe consists of all subrelations of some equivalence relation.

[3]Maddux used the term "pair" to denote the abstract analogue of relations (below the identity) with either one or two ordered pairs. We use it to denote only the analogue of relations with two ordered pairs.

It might be helpful to the reader if we clarify further the relationship between Maddux's representation theorem for pair-dense relation algebras in [10] and the representation theorem of Jónsson-Tarski in [7] for atomic relation algebras in which all atoms are points. Our remarks will also help to illuminate some later statements we make concerning the connections between Maddux's theorem and our own work. It is implicitly proved in Maddux's paper that all *pair-dense* relation algebras—algebras in which the identity element is the sum of points and "pairs"—are subalgebras of atomic pair-dense relation algebras, and all *point-dense* relation algebras—algebras in which the identity element is the sum of points—are subalgebras of atomic point-dense relation algebras with all atoms being points (see Theorem 48, the proof of Theorem 54, and Theorem 40(v).) The latter algebras (for the case of point-density, but not for the case of pair-density) are representable by the Jónsson-Tarski theorem.

The representation theorem for point-dense relation algebras was extended by Maddux to point-dense semi-associative relation algebras (models of Tarski's axioms except that the associative law for relative multiplication is weakened) by showing that such algebras must in fact be relation algebras. He also proved that, for simple pair-dense relation algebras, there are representations which preserve arbitrary Boolean sums as unions. (Such representations are said to be *complete*.)

groups. The penultimate theorem we shall discuss provides an explanation of this connection. Of equal importance, it opens up the prospect of proving a positive representation theorem for all abstract relation algebras—one in which the role of algebras of binary relations is played by relation algebras constructed by means of groups and systems of cosets. Such a theorem, if true, would give a very satisfactory solution to Tarski's representation problem, though one quite different from the solution envisioned by Tarski.

Measurable elements and measurable relation algebras. The property of being a point or a pair below the identity element is expressible in the equational language of relation algebras. Generally, however, the size of an element below the identity—even for small sizes like 3, 4 or 5—is not expressible in the language. To overcome this difficulty we must find another way of expressing size. This leads to the notion of a measurable atom.

For an element x below the identity (a *subidentity element*), the *square* on x (or the *square with side* x) is defined to be the element $x;1;x$. In algebras of binary relations, such squares are just Cartesian squares, that is, relations of the form $Y \times Y$ for some subset Y of the base set U (provided that the universal relation is the Cartesian square $U \times U$). A subidentity atom is said to be *measurable* if its square is the sum of functional elements, and the number of non-zero functional elements is the *measure* or *size* of the atom. If this sum is finite, then the atom is said to be *finitely measurable*. The name comes from the fact that, for algebras of binary relations in which the universal relation is a Cartesian square, the number of non-zero functional elements beneath the square on a subidentity atom is precisely the same as the number of pairs in that atom. For instance, in such an algebra a subidentity atom consists of a single ordered pair just in case its square is a function; it consists of two ordered pairs just in case its square is the sum of two non-empty functions; it consists of three ordered pairs just in case its square is the sum of three non-empty functions; and so on.

A relation algebra is defined to be *measurable* if its identity element is the sum of measurable atoms. If each of these atoms is in fact finitely measurable, then the algebra is said to be *finitely measurable*. Of course the atomic relation algebras with functional atoms of Jónsson and Tarski are measurable, since *every* element is the sum of functional elements. The point-dense and pair-dense relation algebras of Maddux are finitely measurable whenever they are atomic; in fact each subidentity atom has measure one or two.[4] The original goal of the present research was to carry out an analysis

[4]People acquainted with Maddux [10] may notice that we require measurable elements to be atoms, whereas he does not require points and pairs to be atoms. The difference is not essential for elements of finite measure. Given a natural number $n \geq 1$, define a non-zero subidentity element x (atom or not) in a relation algebra to be *n-like* if its square $x;1;x$ is the sum of n functional elements. Points and pairs in the sense of Maddux are precisely 1-like and 2-like elements, respectively. One can prove that: (1) in a simple relation algebra

of measurable relation algebras and to describe them all, up to isomorphism. The project was initiated by the first author and brought to its present state by both authors. (We shall say a few more words about this later.)

Algebras of relations from groups. The first task is the construction of a comprehensive class of examples of measurable relation algebras. To motivate the definition, consider two fixed groups and an isomorphism between quotients of these groups. For example, take the groups to be Z_6 and Z_9, the cyclic groups of order six and nine, and take the isomorphism φ to be the canonical isomorphism between the quotients

$$Z_6/\{0,3\} \quad \text{and} \quad Z_9/\{0,3,6\}.$$

Thus, φ is the following mapping of cosets:

$$\begin{array}{rcl} H_0 = \{0,3\} & \overset{\varphi}{\longmapsto} & K_0 = \{0,3,6\}, \\ H_1 = \{1,4\} & \longmapsto & K_1 = \{1,4,7\}, \\ H_2 = \{2,5\} & \longmapsto & K_2 = \{2,5,8\}. \end{array}$$

Using this correspondence, one can define three binary relations:

$$\begin{aligned} R_0 &= [H_0 \times K_0] \cup [H_1 \times K_1] \cup [H_2 \times K_2] \\ &= \{(a,b) \colon a \in Z_6,\ b \in Z_9 \text{ and } b \equiv a \bmod 3\}\,, \end{aligned}$$

$$\begin{aligned} R_1 &= [H_0 \times (K_0 + K_1)] \cup [H_1 \times (K_1 + K_1)] \cup [H_2 \times (K_2 + K_1)] \\ &= [H_0 \times K_1] \cup [H_1 \times K_2] \cup [H_2 \times K_0] \\ &= \{(a,b) \colon a \in Z_6,\ b \in Z_9 \text{ and } b \equiv a + 1 \bmod 3\}\,, \end{aligned}$$

$$\begin{aligned} R_2 &= [H_0 \times (K_0 + K_2)] \cup [H_1 \times (K_1 + K_2)] \cup [H_2 \times (K_2 + K_2)] \\ &= [H_0 \times K_2] \cup [H_1 \times K_0] \cup [H_2 \times K_1] \\ &= \{(a,b) \colon a \in Z_6,\ b \in Z_9 \text{ and } b \equiv a + 2 \bmod 3\}\,. \end{aligned}$$

The relations R_0, R_1, and R_2 are rather like *representatives* of the cosets H_0, H_1, and H_2, and together they give a representation of Z_3 that has the flavor of the Cayley representation of Z_3.

(a relation algebra with exactly two congruence relations), an n-like element is necessarily the sum of measurable atoms, and the total sum of the measures of these atoms is at most n. Define a relation algebra to be *n-dense* if its identity element is the sum of n-like elements. Point dense and pair dense relation algebras in the sense of Maddux are exactly 1-dense and 2-dense relation algebras, respectively. Using observation (1), it can be shown that every n-dense relation algebra is a subalgebra of an atomic, measurable relation algebra in which all of the subidentity atoms have measure at most n. Thus, the restriction to atoms in our definition of an element of finite measure does not limit the generality of the conclusions we can obtain in our theorems.

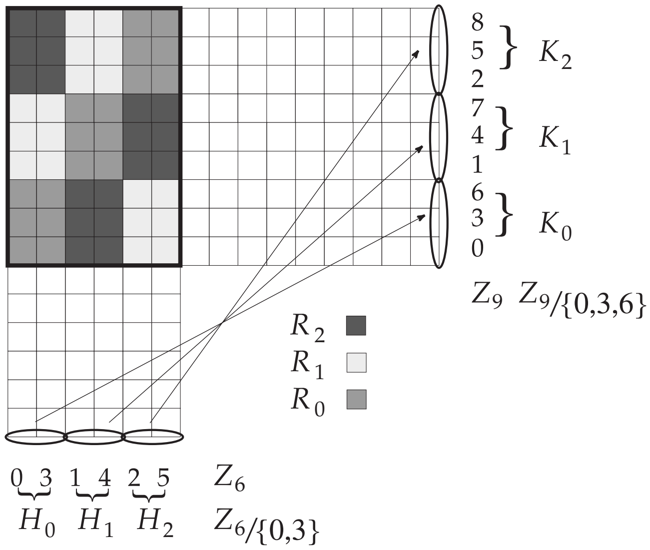

FIGURE 1. The relations R_0, R_1, and R_2.

For the general definition, consider a system

$$G = \langle G_x : x \in I \rangle$$

of pairwise disjoint groups

$$\langle G_x, \circ, ^{-1}, e_x \rangle,$$

and a system

$$\varphi = \langle \varphi_{xy} : \ (x, y) \in \mathcal{E} \rangle$$

of associated isomorphisms between quotients of these groups:

$$\varphi_{xy} : G_x/H_{xy} \longmapsto G_y/K_{xy}$$

(where $\mathcal{E}$ is some equivalence relation on the group index set I).[5] Suppose that

$$\varphi_{xy}(H_{xy,\alpha}) = K_{xy,\alpha}$$

(where $\langle H_{xy,\alpha} : \alpha < \kappa_{xy} \rangle$ and $\langle K_{xy,\alpha} : \alpha < \kappa_{xy} \rangle$ are coset systems for H_{xy} and K_{xy} indexed by some ordinal κ_{xy}, and $H_{xy,0}$ is the identity coset). The pair (G, φ) is called a *group pair*.

Define a binary relation

$$R_{xy,\alpha} = \bigcup_{\gamma<\kappa_{xy}} H_{xy,\gamma} \times \varphi_{xy}[H_{xy,\gamma} \circ H_{xy,\alpha}] = \bigcup_{\gamma<\kappa_{xy}} H_{xy,\gamma} \times (K_{xy,\gamma} \circ K_{xy,\alpha})$$

[5]To avoid cumbersome notation, we have used the same symbol to denote a group and its universe. Notice that the superscript "-1" is used to denote a group inverse operation as well as the operation of forming the inverse of a relation or the inverse of a function. This multiple usage is traditional, and should not cause any confusion.

for each pair (x, y) in $\mathcal{E}$ and each index $\alpha < \kappa_{xy}$. The relation $R_{xy,\alpha}$ *represents* the coset $H_{xy,\alpha}$ in much the same way that the set of pairs

$$\{(g, g \circ f)\colon g \in G_x\} = \bigcup_{f \in G_x} \{g\} \times \{g \circ f\}$$

represents the group element f in the Cayley representation of the group G_x.

The relations $R_{xy,\alpha}$ are pairwise disjoint (for distinct α) and have $G_x \times G_y$ as their union. Therefore, the collection A of all possible unions of these relations (where (x, y) varies over all pairs in $\mathcal{E}$, and α over indices less than κ_{xy}) forms an atomic Boolean algebra of binary relations (on the base set $U = \bigcup_{x \in I} G_x$) that is closed under arbitrary unions and intersections. The universal relation (the Boolean unit) of the algebra is the binary relation

$$E = \bigcup \{G_x \times G_y\colon\ (x, y) \in \mathcal{E}\},$$

and the atoms are just the relations $R_{xy,\alpha}$ for distinct pairs (x, y) in $\mathcal{E}$ and distinct indices $\alpha < \kappa_{xy}$. The set A is not, in general, the universe of a set relation algebra: it does not contain the identity relation on U, nor is it closed under relational composition and inversion. The first theorem formulates conditions on the group pair that ensure we end up with a set relation algebra.

THEOREM 1. (i) *A contains the identity relation on the set $\bigcup_{x \in I} G_x$ if and only if φ_{xx} is the identity automorphism of $G_x/\{e_x\}$ for each x in I.*

(ii) *A is closed under formation of inverses if and only if*

$$\varphi_{yx} = \varphi_{xy}^{-1}$$

for every pair (x, y) in $\mathcal{E}$.

(iii) *A is closed under relational composition if and only if*

$$\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}$$

and

$$\hat{\varphi}_{xy} \mid \hat{\varphi}_{yz} = \hat{\varphi}_{xz}$$

for all pairs (x, y) and (y, z) in $\mathcal{E}$.

The condition of (iii) needs a word of explanation. We are given the mappings

$$G_x/H_{xy} \quad \overset{\varphi_{xy}}{\longmapsto} \quad G_y/K_{xy}, \qquad G_y/H_{yz} \quad \overset{\varphi_{yz}}{\longmapsto} \quad G_z/K_{yz},$$

$$G_x/H_{xz} \quad \overset{\varphi_{xz}}{\longmapsto} \quad G_z/K_{xz}.$$

The subgroup K_{xy} may not coincide with the subgroup H_{yz} at all, so it is not meaningful to speak about the composition of φ_{xy} and φ_{yz}. In order to be able to compose quotient isomorphisms, one has first to form a common quotient group using the complex product of the subgroups, and then compose the induced isomorphisms $\hat{\varphi}_{xy}$ and $\hat{\varphi}_{yz}$, as is done in the condition of (iii):

$$\begin{tikzcd} G_x/(H_{xy} \circ H_{xz}) \arrow[r, mapsto, "\hat{\varphi}_{xy}"] \arrow[rr, mapsto, bend right, "\hat{\varphi}_{xz}"'] & G_y/(K_{xy} \circ H_{yz}) \arrow[r, mapsto, "\hat{\varphi}_{yz}"] & G_z/(K_{xz} \circ K_{yz}). \end{tikzcd}$$

If a pair (G, φ) satisfies the three conditions of Theorem 1, then there are explicit formulas for computing the identity relation, the inverse of an atomic relation, and the composition of two atomic relations. For instance, the identity relation is the union of the identity relations on the groups G_x:

$$Id = \bigcup\{R_{xx,0} : x \in I\}.$$

The inverse of an atomic relation $R_{xy,\alpha}$ is obtained using the inverse of the corresponding coset:

$$R^{-1}_{xy,\alpha} = R_{yx,\beta}, \quad \text{where} \quad H^{-1}_{xy,\alpha} = H_{xy,\beta}.$$

The composition of two atomic relations is determined by the formulas

$$R_{xy,\alpha} \mid R_{wz,\beta} = \varnothing \qquad \text{if} \quad y \neq w,$$

and

$$R_{xy,\alpha} \mid R_{yz,\beta} = \bigcup \left\{ R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi^{-1}_{xy}[K_{xy,\alpha} \circ H_{yz,\beta}] \right\}.$$

Another way of expressing the last equation is:

$$R_{xz,\gamma} \subseteq R_{xy,\alpha} \mid R_{yz,\beta} \quad \text{if and only if} \quad H_{xz,\gamma} \subseteq \varphi^{-1}_{xy}[K_{xy,\alpha} \circ H_{yz,\beta}].$$

This formula shows that, in some sense, the composition of the two relations $R_{xy,\alpha}$ and $R_{yz,\beta}$ represents the complex product of the two cosets $H_{xy,\alpha}$ and $H_{yz,\beta}$. Since these cosets "live" in disjoint groups, namely G_x and G_y respectively, it is necessary to translate the coset $H_{xy,\alpha}$ over to the group G_y first, in order to be able to form its complex product with $H_{yz,\beta}$. That is why the translation $K_{xy,\alpha}$ is used. Because the composite relation is a subset of $G_x \times G_z$, it is then necessary to translate the complex product of the cosets back to G_x using the isomorphism φ^{-1}_{xy} in order to write the product as a coset of $H_{xy} \circ H_{xz}$.

Theorem 1 motivates the following definition.

DEFINITION 1. A group pair (G, φ) is called a *(group) frame* just in case it satisfies the following four conditions whenever each of the relevant pairs is in $\mathcal{E}$:

(i) φ_{xx} is the identity automorphism of $G_x/\{e_x\}$;
(ii) φ_{yx} is the inverse of φ_{xy} ;
(iii) $\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}$;
(iv) $\hat{\varphi}_{xy} \mid \hat{\varphi}_{yz} = \hat{\varphi}_{xz}$.

Notice that the second condition implies

$$K_{yx} = H_{xy} \quad \text{and} \quad H_{yx} = K_{xy}.$$

Consequently, it follows from the third condition that

$$\varphi_{yz}[K_{xy} \circ H_{yz}] = K_{xz} \circ K_{yz} = \varphi_{xz}[H_{xy} \circ H_{xz}].$$

Theorem 1 can be reformulated as saying that A is the universe of a set relation algebra just in case the pair $\mathcal{F} = (G, \varphi)$ is a frame. When this is the case, we denote the corresponding set relation algebra by $\mathfrak{G}[\mathcal{F}]$ and call it the (full) *group relation algebra on* $\mathcal{F}$.[6] It is always a measurable relation algebra. Its subidentity atoms are the relations

$$R_{xx,0} = \{(g, g) \colon g \in G_x\} = \{(g, g \circ e_x) \colon g \in G_x\},$$

for x in I. The square

$$R_{xx,0} \mid E \mid R_{xx,0}$$

coincides with $G_x \times G_x$, and is the disjoint union of the functions

$$R_{xx,\alpha} = \{(g, g \circ f_\alpha) \colon g \in G_x\}$$

(where $\langle \{f_\alpha\} \colon \alpha < \kappa_{xx} \rangle$ is the assumed enumeration of the cosets of the subgroup $\{e_x\}$ in G_x, and f_0 is e_x, and E is the universal relation.) The number of these functions included in the union is exactly the same as the number of pairs in $R_{xx,0}$ and the number of elements in G_x. Thus, the number of these functions is a measure of the size of the relation $R_{xx,0}$. Notice, in passing, that the mapping taking each group element f_α to the relation $R_{xx,\alpha}$ is just the Cayley representation of G_x.

Examples of group relation algebras. How does one go about constructing a frame? What do group relation algebras look like in concrete cases? Perhaps the construction is easiest to understand in the context of a simple example, a kind of "power" construction. Fix a group M and a normal subgroup N. For each element x in a given index set I, let G_x be an isomorphic copy of M (chosen so that distinct copies are pairwise disjoint) and ψ_x an isomorphism from the quotient group M/N to the corresponding quotient group of G_x. Take the mapping φ_{xy} to be the natural isomorphism between the quotient groups of G_x and of G_y,

$$\varphi_{xy} = \psi_x^{-1} \mid \psi_y,$$

[6] Usually, the term *group relation algebra* is reserved for a relation algebra obtained as the algebra of complexes (subsets) of a single group. (It was J. C. C. McKinsey who first pointed out that this algebra of complexes is indeed a relation algebra; see Jónsson-Tarski [5].) However, the algebra of complexes of a group M is canonically isomorphic (via the Cayley representation) to the very special case of our construction when the system G consists of the single group M. In view of this, it seems justified to extend the scope of the phrase *group relation algebra* to our more general construction.

for distinct x and y in I. Thus, these mappings are all isomorphisms between copies of the single quotient group M/N. Take φ_{xx} to be the appropriate identity isomorphism, as required by the definition of a frame.

The resulting pair $\mathcal{F} = (G, \varphi)$ is a frame, and the corresponding group relation algebra $\mathfrak{G}[\mathcal{F}]$ is an algebra of binary relations. If we take the indices α of the atomic relations in $\mathfrak{G}[\mathcal{F}]$ to be the corresponding cosets of M/N (instead of ordinals), then there are especially simple formulas for computing the converse of an atomic relation and the composition of two atomic relations in $\mathfrak{G}[\mathcal{F}]$:

$$R_{xy,\alpha}^{-1} = R_{yx,\alpha^{-1}} \quad \text{and} \quad R_{xy,\alpha} \mid R_{yz,\beta} = R_{xz,\alpha\circ\beta}$$

when x, y, and z are distinct elements of I. (Here α^{-1} denotes the inverse of the coset α, and $\alpha \circ \beta$ the product of the cosets α and β, in M/N.)

When N is the trivial (one-element) subgroup of M, each atomic relation $R_{xy,\alpha}$ is a function and in fact a bijection from G_x to G_y. In this case, $\mathfrak{G}[\mathcal{F}]$ is an example of an atomic relation algebra with functional atoms, as in the first of the Jónsson-Tarski representation theorems (see Figure 2). At the other extreme, when N coincides with M, there is only one atomic relation,

$$R_{xy,0} = G_x \times G_y,$$

for each pair of distinct indices x, y in I (see Figure 4). In general, if the normal subgroup N has order λ and index κ in M (that is, if N contains λ elements and has κ cosets in M), then there will be κ distinct atomic relations of the form $R_{xy,\alpha}$, and each of them will be the union of λ pairwise disjoint bijections from G_x to G_y (see Figure 3).

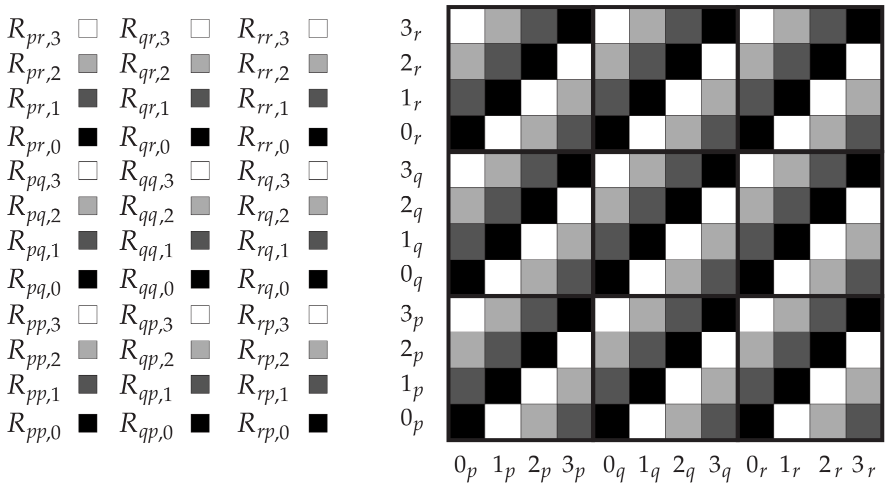

FIGURE 2. The power construction for the group Z_4 and the subgroup $\{0\}$, where $I = \{p, q, r\}$ and $\mathcal{E} = I \times I$.

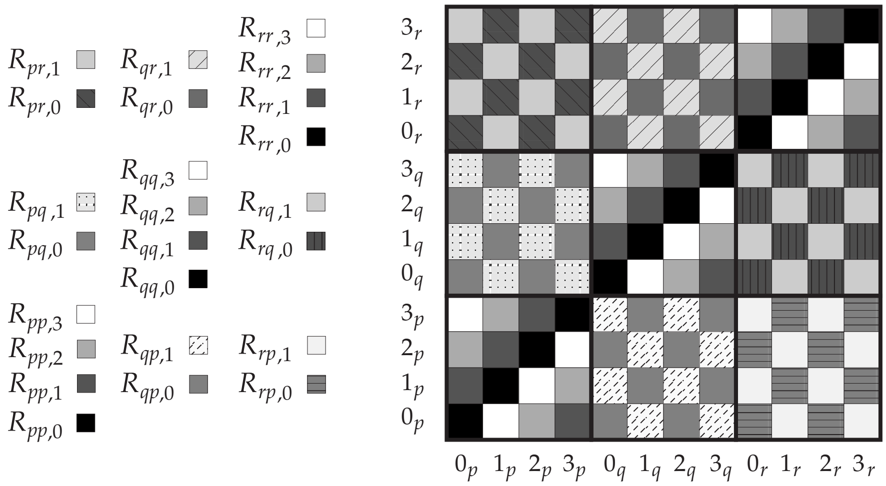

FIGURE 3. The power construction for the group Z_4 and the subgroup Z_2, where $I = \{p, q, r\}$ and $\mathcal{E} = I \times I$.

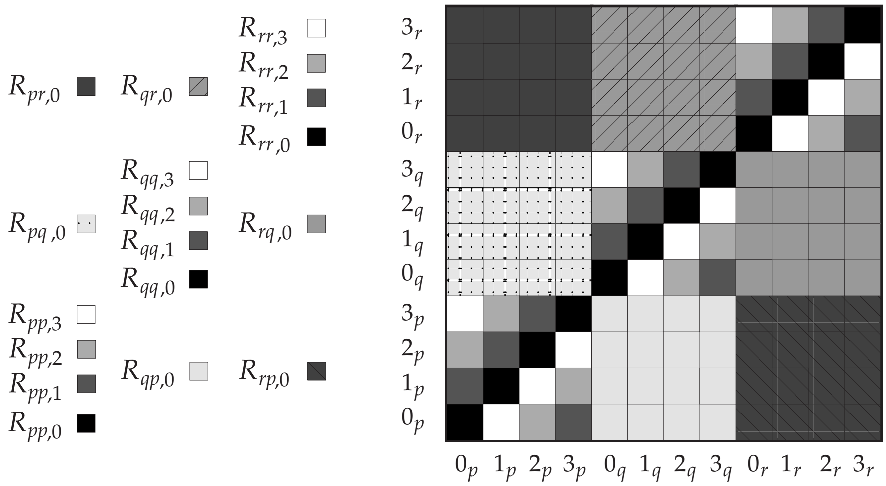

FIGURE 4. The power construction for the group Z_4 and the subgroup Z_4, where $I = \{p, q, r\}$ and $\mathcal{E} = I \times I$.

(In the general case of the power construction, $\mathcal{E}$ is allowed to be an arbitrary equivalence relation on I. Moreover, the group M and normal subgroup N are fixed for a given equivalence class of $\mathcal{E}$, but different equivalence classes may have different groups and normal subgroups.)

The most trivial case of the power construction is when the fixed group M is the one-element group. In this case, $\mathfrak{G}[\mathcal{F}]$ is just the full set relation algebra on the base set $U = \bigcup_{x \in I} G_x$. Moreover, every full set relation algebra can be obtained as a group relation algebra in this fashion. Thus, the construction of full set relation algebras may be viewed as the most trivial case of the construction of group relation algebras, namely the case when all the groups have order one.

It follows from this observation that the class of algebras embeddable into group relation algebras coincides with the class of representable relation algebras. (In particular, the class is equationally axiomatizable, by the aforementioned theorem of Tarski [18].) However, the description of representable relation algebras in terms of group relation algebras seems much more advantageous, because the class of full group relation algebras is substantially more varied and interesting than the class of full set relation algebras. (See Corollaries 1 and 2 below.)

A second example of the group relation algebra construction that is easy to describe is the one in which all of the groups are cyclic. Suppose $G = \langle G_x : x \in I \rangle$ is a family of (pairwise disjoint) cyclic groups and $\mathcal{E}$ an equivalence relation on I. To avoid unnecessary complications in notation, we consider here only the case when the groups are finite. Fix a generator g_x of each group G_x. Let $\langle \kappa_{xy} : (x, y) \in \mathcal{E} \rangle$ be a system of positive integers satisfying the following conditions for all appropriate pairs in $\mathcal{E}$:

(i) κ_{xy} is a common divisor of the orders of G_x and G_y,
(ii) κ_{xx} is equal to the order of G_x,
(iii) $\kappa_{yx} = \kappa_{xy}$,
(iv) $\gcd(\kappa_{xy}, \kappa_{yz}) = \gcd(\kappa_{xy}, \kappa_{xz}) = \gcd(\kappa_{xz}, \kappa_{yz})$.

Condition (i) ensures that there are (unique) subgroups H_{xy} and K_{xy} of index κ_{xy} in G_x and G_y respectively. (In other words, these subgroups have κ_{xy} many cosets.) Therefore, the quotient groups G_x/H_{xy} and G_y/K_{xy} are isomorphic, and in fact there is a (uniquely determined) isomorphism φ_{xy} between them that maps the generator g_x/H_{xy} to the generator g_y/K_{xy}. Conditions (ii) and (iii), and the definition of the quotient isomorphisms, ensure that frame conditions (i) and (ii) (in Definition 1) are satisfied. The complex product $H_{xy} \circ H_{xz}$ is a subgroup of G_x of index $d = \gcd(\kappa_{xy}, \kappa_{xz})$. Condition (iv) says that the complex products $K_{xy} \circ H_{yz}$ and $K_{xz} \circ K_{yz}$ also have index d. This, together with the definition of the quotient isomorphisms, ensures that frame conditions (iii) and (iv) are satisfied. Thus, the pair $\mathcal{F} = (G, \varphi)$ is a frame.

If every group in $\mathcal{F}$ has order one or two, then the group relation algebra $\mathfrak{G}[\mathcal{F}]$ is an example of a pair-dense relation algebra in the sense of Maddux [10]. When $\kappa_{xy} = 2$, there are exactly two relations: $R_{xy,0}$ and $R_{xy,1}$. Each of them is a function (and in fact a bijection from G_x to G_y) with exactly two pairs. When $\kappa_{xy} = 1$, there is only one relation: $R_{xy,0} = G_x \times G_y$. It contains either four pairs, two pairs, or one pair, according as both groups G_x and G_y have order two, exactly one of these groups has order two and the other order one, or both groups have order one. (See Figure 5.)

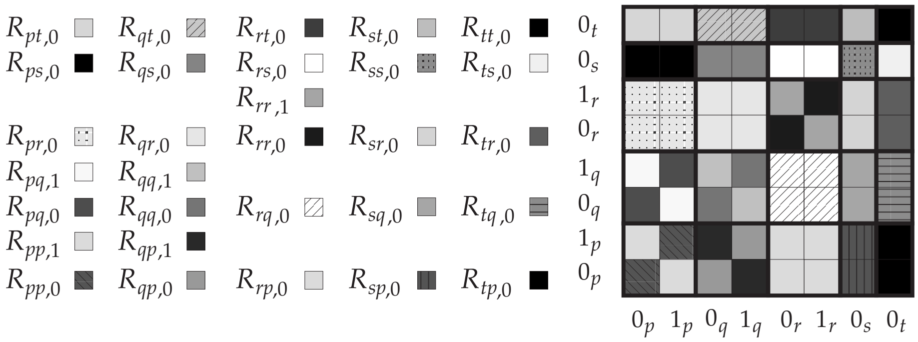

FIGURE 5. A pair-dense group relation algebra with groups G_p, G_q and G_r of order two, and groups G_s and G_t of order one, where $I = \{p, q, r, s, t\}$ and $\mathcal{E} = I \times I$. Also, $\kappa_{pq} = \kappa_{qp} = 2$, and $\kappa_{xy} = 1$ for all other pairs (x, y) in $\mathcal{E}$ of distinct elements.

The main problem. We have described a broad class of examples of measurable relation algebras: the group relation algebras. Is it possible that every atomic, measurable relation algebra is essentially isomorphic to a group relation algebra? To make this question precise, we have to understand what is meant by "essentially isomorphic". An atomic relation algebra—and in particular, an atomic, measurable relation algebra—need not be complete as a Boolean algebra. There may be infinite subsets of the universe whose (Boolean) sum does not exist, that is, there may be infinite subsets that have no least upper bound in the algebra. On the other hand, full group relation algebras are always complete: the union of every collection of relations in the algebra is again a relation in the algebra. For this reason, there certainly exist atomic, measurable relation algebras that are not isomorphic to group relation algebras.

Incompleteness is a relatively minor defect, one that is correctable by "filling in" all of the missing infinite sums. The technical way of doing this is to pass to what is called the *completion* of the relation algebra. Every relation algebra $\mathfrak{A}$ (whether atomic or not) has a unique (MacNeille-Dedekind) completion (see [12]). It is the minimal relation algebraic extension of $\mathfrak{A}$ that is complete in the sense that all sums and products of infinite sets of elements exist. If a sum or product of a set of elements of $\mathfrak{A}$ exists in $\mathfrak{A}$, then this set will have the same sum or product in the completion. (In particular, a complete relation algebra is its own completion.) Most importantly, every element in the completion is the sum of a set of elements *of* $\mathfrak{A}$. In particular, $\mathfrak{A}$ and its completion have exactly the same set of atoms. If $\mathfrak{A}$ is measurable, then its completion has exactly the same subidentity atoms, and the squares on these atoms are the sums of exactly the same sets of functional elements.

Consequently, the completion of an atomic, measurable relation algebra is again atomic and measurable.

We shall say that two relation algebras are *essentially isomorphic* if their completions are isomorphic. The principal problem with which we are concerned can now be formulated as follows.

PROBLEM 1. *Is every atomic, measurable relation algebra essentially isomorphic to a group relation algebra*?

The hypothesis of atomicity is actually superfluous in the case of finitely measurable relation algebras, as the following theorem shows. (Note that a finitely measurable relation algebra need not be finite.)

THEOREM 2. *Every finitely measurable relation algebra is atomic.*

Problem 1 asks whether group relation algebras exhaust the examples of atomic, measurable relation algebras. It turns out that, in general, the answer is negative. However, many interesting special cases of the problem do have positive solutions. Most importantly, a somewhat modified form of the problem turns out to have a completely positive solution.

The structure of atomic, measurable relation algebras. Atomic, measurable relation algebras have a highly structured local behavior that reflects many properties of groups. In order to establish a representation theorem, it is necessary to analyze this local behavior. We summarize the salient points of the analysis.

Consider an atomic, measurable relation algebra

$$\mathfrak{A} = \langle A, +, -, ;, {}^{\smile}, 1\text{'} \rangle.$$

Take I to be the set of measurable atoms, and $\mathcal{E}$ to be the equivalence relation on I that holds between two measurable atoms x and y just in case $1;x;1 = 1;y;1$ (or, equivalently, just in case $x;1;y \neq 0$). For each measurable atom x, the functional elements below the square $x;1;x$ form a group G_x under the operations of relative multiplication and conversion; the identity element of the group is x. We shall call $G = \langle G_x : x \in I \rangle$ the system of *associated groups*.

For two measurable atoms x and y (with (x, y) in $\mathcal{E}$), an atom a below $x;1;y$ determines normal subgroups of G_x and G_y:

$$H_{xy} = \{f \in G_x : f;a = a\} \quad \text{and} \quad K_{xy} = \{g \in G_y : a;g = a\}.$$

These subgroups are independent of the choice of the atom. The atom a also determines an isomorphism φ_a between the quotient groups

$$G_x/H_{xy} \quad \text{and} \quad G_y/K_{xy}$$

which takes a coset H_γ to a coset K_δ just in case

$$H_\gamma;a = a;K_\delta.$$

(The atoms below $x;1;y$ are precisely the elements that can be written in the form $f;a$ for some element f in G_x. Two such atoms $f;a$ and $f';a$ are equal just in case f and f' are in the same coset H_γ. Therefore, it is meaningful to write "$H_\gamma;a$" to denote the atom $f;a$. A similar remark applies to the elements of the form $a;g$ and to the notation $a;K_\delta$.) The quotient isomorphism φ_a determined by a is *not* independent of the choice of the atom. To construct from these quotient isomorphisms a system that satisfies the frame conditions, it is necessary to choose a coordinated system of atoms.

DEFINITION 2. A *semi-scaffold* is a system $\langle a_{xy} : (x,y) \in \mathcal{E}\rangle$ such that a_{xy} is an atom below $x;1;y$ for each pair (x,y) in $\mathcal{E}$, and the following two conditions are satisfied for all appropriate pairs in $\mathcal{E}$:

(i) $a_{xx} = x$,
(ii) $a_{yx} = a_{xy}^{\smile}$.

A semi-scaffold is called a *scaffold* if, in addition,

(iii) $a_{xz} \leq a_{xy};a_{yz}$

holds for all appropriate pairs in $\mathcal{E}$.

The existence of a semi-scaffold in a measurable relation algebra implies that the algebra is atomic. Therefore, no special hypothesis of atomicity is needed. Once a semi-scaffold $a = \langle a_{xy} : (x,y) \in \mathcal{E}\rangle$ has been constructed, the isomorphism φ_{xy} can be unambiguously specified as the quotient isomorphism determined by the atom a_{xy}. We shall refer to $\varphi = \langle \varphi_{xy} : (x,y) \in \mathcal{E}\rangle$ as the system of quotient isomorphisms *associated* with the semi-scaffold, and to $\mathcal{F} = (G, \varphi)$ as the *associated* group pair. The isomorphisms (but not the groups and normal subgroups) are dependent on the choice of the semi-scaffold.

THEOREM 3 (Scaffold Representation Theorem). *If a measurable relation algebra $\mathfrak{A}$ has a scaffold, then the associated group pair $\mathcal{F}$ is a frame, and $\mathfrak{A}$ is essentially isomorphic to the group relation algebra $\mathfrak{G}[\mathcal{F}]$.*

To determine, then, whether a measurable relation algebra is essentially isomorphic to a group relation algebra, it suffices to determine whether the algebra has a scaffold. It is not obvious that this condition is also necessary, but for algebras with representations that preserve arbitrary sums as unions—so-called *complete* representations—it is.

THEOREM 4. *If a measurable relation algebra $\mathfrak{A}$ has a complete representation, then it has a scaffold. Consequently, it must be essentially isomorphic to a group relation algebra.*

Notice that any representation of a finite relation algebra must preserve all sums as unions. Therefore, if a finite, measurable relation algebra has any representation, it must have a scaffold. Another consequence of Theorem 4 is that the answer to Problem 1 is positive in the case of relation algebras with complete representations. This suggests the following reformulation of the problem: does every atomic, measurable relation algebra have a complete representation? Summarizing, an atomic, measurable relation algebra has a complete representation if and only if it is essentially isomorphic to a group relation algebra, and this happens if and only if it has a scaffold.

Group relation algebra representation theorems. There are several interesting cases in which it is not difficult to construct a scaffold.

THEOREM 5. *Let $\mathfrak{A}$ be an atomic, measurable relation algebra. If the associated normal subgroups satisfy the condition*

$$K_{xy} = H_{yz}$$

for all pairs (x, y) and (y, z) in $\mathcal{E}$ with x, y, z pairwise distinct, then $\mathfrak{A}$ has a scaffold. Consequently, $\mathfrak{A}$ is essentially isomorphic to the group relation algebra on the associated frame.

This theorem can be used to describe all atomic relation algebras with functional atoms—the algebras that figure in the first of the Jónsson-Tarski representation theorems. Any such algebra is measurable, so there are systems of associated groups and normal subgroups. It is not difficult to check that the associated normal subgroups satisfy the condition

$$K_{xy} = \{y\} = H_{yz}$$

for all pairs (x, y) and (y, z) in $\mathcal{E}$. Therefore, by Theorem 5, the algebra has a scaffold and is essentially isomorphic to the group relation algebra on the associated frame. Notice that φ_{xy} is an isomorphism from $G_x/\{x\}$ to $G_y/\{y\}$. Thus, the groups G_x and G_y are isomorphic to one another whenever (x, y) is in $\mathcal{E}$, and the associated normal subgroups are trivial.

COROLLARY 1. *A relation algebra is atomic with functional atoms if and only if it is essentially isomorphic to a group relation algebra on a frame in which the groups indexed by elements of the same equivalence class are all isomorphic, and the associated normal subgroups are all trivial.*

The preceding corollary makes clear that Theorem 5 is a substantial generalization of the Jónsson-Tarski representation theorem for atomic relation algebras with functional atoms. The point of Corollary 1 is that it clarifies the structure of the representing set relation algebras that are constructed in the proof of the Jónsson-Tarski theorem: they are essentially the group relation algebras that arise from the special case of the power construction when the normal subgroups are all trivial.

Another case in which it is not difficult to construct a scaffold is when each of the associated groups has order at most two. As a result, we obtain from Theorem 3 the following characterization of Maddux's pair-dense relation algebras in the atomic case.

COROLLARY 2. *An atomic relation algebra is pair-dense if and only if it is essentially isomorphic to the group relation algebra on a frame in which all groups have order at most two.*

It follows from the proof of the final theorem in Maddux's paper that every pair-dense relation algebra is a subalgebra of an atomic, pair-dense relation algebra (see Footnote 3), and hence, by Corollary 2, is embeddable in a group relation algebra in which all the groups have order at most two.

It is worth mentioning two other cases when the construction of a scaffold in an atomic, measurable relation algebra is not difficult: (i) when there are at most four measurable atoms; (ii) when, in each of the associated groups, chains of proper normal subgroups have length at most one (this includes the special case when the associated groups are all simple).[7]

With two exceptions, the results formulated so far were essentially obtained by the first author, before Andréka joined the project. The two exceptions are Theorem 4 and the construction, sketched above, of a group relation algebra on any system of cyclic groups. These two results, and all of the theorems discussed below, were obtained later, as part of our joint collaboration.

THEOREM 6. *If each of the associated groups in an atomic, measurable relation algebra is either cyclic, or else the product of two cyclic groups, then the algebra has a scaffold, and is therefore essentially isomorphic to the group relation algebra on the associated frame.*

As is indicated by Corollary 2, the preceding theorem may be viewed as a generalization of Maddux's representation theorem for pair-dense relation algebras, at least as concerns atomic algebras. (The key case of Maddux's theorem—the atomic one—implicitly assumes that all the groups have order at most two, while Theorem 6 allows arbitrary cyclic groups, and even products of two cyclic groups.)

A non-representable, measurable relation algebra. The next natural step in trying to obtain a positive solution to Problem 1 is to extend the previous theorem to the case when the associated groups are products of at most three cyclic groups. However, it is just at this point that things break down. In fact, there exists a finite, measurable relation algebra with five measurable atoms that is not representable at all. The groups in this example are products of three cyclic groups.

To construct such an algebra, begin by constructing a group relation algebra on a frame

$$\mathcal{F} = (\langle G_x : x \in I\rangle, \langle \varphi_{xy} : (x, y) \in I \times I\rangle)$$

[7] It is a consequence of (ii) that 7-dense relation algebras (defined in Footnote 4) are always representable. It follows from Theorem 7 that there are 8-dense relation algebras which are not representable.

in which the index set I has five elements, say

$$I = \{p, q, r, s, t\}.$$

Each of the groups G_x is a copy of $Z_2 \times Z_2 \times Z_2$, where $Z_2 = \{0, 1\}$ denotes the cyclic group of order two.

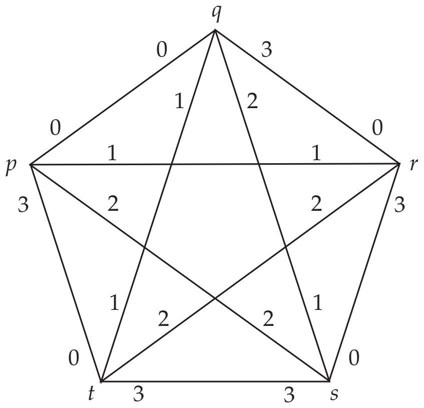

FIGURE 6. Normal subgroup diagram.

To describe the subgroups H_{xy} and K_{xy} for distinct x and y, consider the following four subgroups of $Z_2 \times Z_2 \times Z_2$:

$$L_0 = Z_2 \times \{0\} \times \{0\}, \quad L_1 = \{0\} \times Z_2 \times \{0\},$$
$$L_2 = \{0\} \times \{0\} \times Z_2, \quad L_3 = \{(0,0,0), (1,1,1)\}.$$

Take H_{xy}, respectively K_{xy}, to be the copy of one of these four subgroups in G_x, respectively G_y, according to the prescription given in Figure 6. For example, the subgroup H_{pt} is the copy of L_3 in G_p and the subgroup K_{pt} is the copy of L_0 in G_t, since the edge between the vertices p and t in the diagram is labeled with 3 and 0. Similarly, the subgroup H_{qs} is the copy of L_2 in G_q and the subgroup K_{qs} is the copy of L_1 in G_s, since the edge from q to s is labeled with 2 and 1.

The quotient isomorphisms φ_{xy}, for distinct x and y, are completely determined by the requirement that $\hat{\varphi}_{xy} \mid \hat{\varphi}_{yz} = \hat{\varphi}_{xz}$. For instance, according to the diagram, we must have

$$\varphi_{pq}[L_0 \circ L_3] = L_0 \circ L_1, \quad \varphi_{pq}[L_0 \circ L_1] = L_0 \circ L_3, \quad \varphi_{pq}[L_0 \circ L_2] = L_0 \circ L_2$$

(see Figure 7(a), (b), and (c) respectively). (The composite subgroups inside the brackets should actually be interpreted as denoting their copies in G_p, and the composite subgroups on the right-hand sides of the equality signs should be interpreted as denoting their copies in G_q.) These three requirements determine φ_{pq}. The mappings φ_{xx} are of course the appropriate identity automorphisms, as required by frame condition (i).

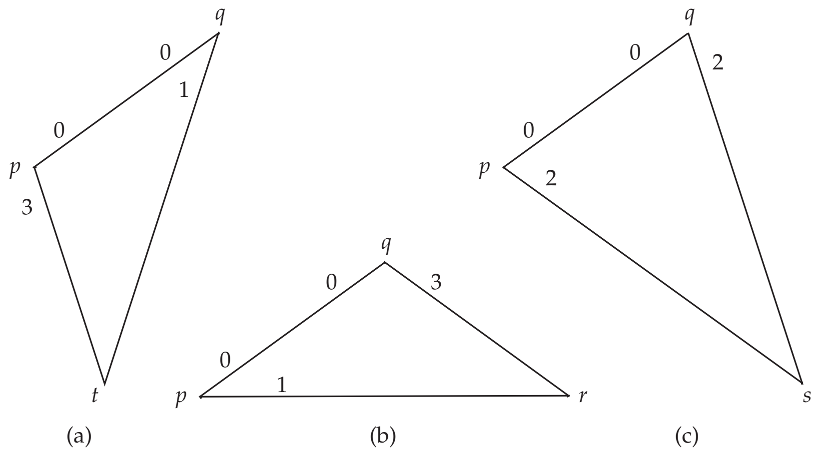

FIGURE 7. The triangles from the pentagon that determine φ_{pq}.

The resulting group pair $\mathcal{F}$ is a frame, so we can construct the group relation algebra $\mathfrak{G}[\mathcal{F}]$. Create a new algebra $\mathfrak{A}$ by modifying the relational composition operation of $\mathfrak{G}[\mathcal{F}]$. Thus, the algebra $\mathfrak{A}$ has the same universe as $\mathfrak{G}[\mathcal{F}]$, and the same operations as $\mathfrak{G}[\mathcal{F}]$, except for relative multiplication. The operation of relative multiplication in $\mathfrak{A}$, which we denote by $\otimes$, is defined on atomic relations as follows:

$$R_{xy,\alpha} \otimes R_{wz,\beta} = R_{xy,\alpha} \mid R_{wz,\beta}$$

whenever $y \neq w$ or $\{x, y, z\} \neq \{p, q, r\}$, and

$$R_{xy,\alpha} \otimes R_{yz,\beta} = G_x \times G_y \sim (R_{xy,\alpha} \mid R_{yz,\beta})$$

whenever $\{x, y, z\} = \{p, q, r\}$. The operation is extended to arbitrary relations by making it distributive over unions. Thus, the operation of relative multiplication in $\mathfrak{A}$ is obtained by changing only slightly the operation of relational composition in $\mathfrak{G}[\mathcal{F}]$ as it affects atomic relations: namely, for those pairs of atomic relations that are indexed, in some order, by the vertices of the triangle $\triangle pqr$, their relative product has been shifted to the complement of what it is in $\mathfrak{G}[\mathcal{F}]$. (See Figure 8.)

THEOREM 7. *The algebra $\mathfrak{A}$ is a finite, measurable relation algebra that is not representable.*

The proof of non-representability proceeds by showing that $\mathfrak{A}$ does not have a scaffold, and then invoking Theorem 4. The key observations in the proof are consequences of the lattice-theoretic properties of the subgroups L_0, L_1, L_2, and L_3 in the lattice of subgroups of $Z_2 \times Z_2 \times Z_2$: the meet (intersection) of any two of these subgroups is the trivial subgroup; the join (complex product) of any two of them is incomparable with the join of

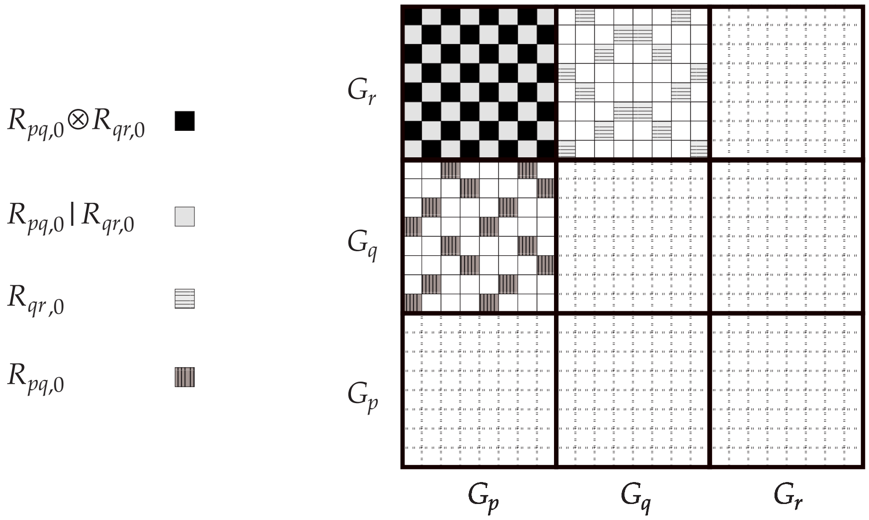

FIGURE 8. A comparison of the relative multiplication and relational composition operations in a sector of the non-representable, measurable relation algebra $\mathfrak{A}$.

any other two of them; and the join of any three of them is always the top element.

The group Z_2 can be replaced everywhere in the preceding construction by an arbitrary non-trivial Abelian group. The mappings φ_{xy} are no longer uniquely determined, and the definition of relative multiplication is slightly more involved (see below). In each case we get an atomic, measurable relation algebra that is not representable. These are new examples of non-representable relation algebras, with a completely different underlying motivation than the examples that have appeared so far in the literature.

Coset relation algebras. The preceding example makes clear that the class of group relation algebras is much narrower than the class of complete, atomic, measurable relation algebras. Yet the structure of these latter algebras is amazingly close to that of group relation algebras. This similarity made us suspect the existence of a more general class of algebras constructed from groups that *would* coincide with the class of all complete, atomic, measurable relation algebras. Eventually, we found it. The missing ingredient was a system of cosets of the composite subgroups that could be used to "shift" the operation of relational composition.

Consider, once more, a system $G = \langle G_x : x \in I \rangle$ of pairwise disjoint groups and an associated system $\varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle$ of quotient isomorphisms. Introduce a system

$$C = \langle C_{xyz} : (x, y, z) \in \mathcal{E}_3 \rangle$$

of "shifting" cosets: C_{xyz} is a coset of the composite subgroup $H_{xy} \circ H_{xz}$ (in G_x) for each triple (x, y, z) in $\mathcal{E}_3$. Here, $\mathcal{E}_3$ denotes the set of triples (x, y, z) such that (x, y) and (y, z) are in $\mathcal{E}$.

Define the binary relations $R_{xy,\alpha}$ exactly as before, and take A to be the set of all possible unions of these relations, just as before. Parts (i) and (ii) of Theorem 1 continue to be valid. With the help of the coset system C, define a new binary operation $\otimes$ between the atomic relations of A:

$$R_{xy,\alpha} \otimes R_{wz,\beta} = \varnothing \qquad \text{if} \quad y \neq w,$$

and

$$R_{xy,\alpha} \otimes R_{yz,\beta} = \bigcup \left\{ R_{xz,\gamma} \colon H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}] \circ C_{xyz} \right\}.$$

Extend $\otimes$ to an operation on arbitrary relations in A by requiring it to be distributive over unions. Notice the very close analogy between the definition of $\otimes$ and the value of $R_{xy,\alpha} \mid R_{yz,\beta}$ when condition (iii) of Theorem 1 is satisfied. The former differs from the latter only in the occurrence of the coset C_{xyz}. In some sense, coset multiplication has been used to "shift" the value of the composition of the two atomic relations. The new product still represents a coset of $H_{xy} \circ H_{xz}$, but not the one corresponding to $K_{xy,\alpha} \circ H_{yz,\beta}$.

When will A be the universe of a relation algebra? It turns out that two kinds of conditions are required. The first are general conditions, similar to the frame conditions, that govern the relationship between the various quotient isomorphisms. In fact, the only frame condition that must be modified is the fourth one, in order to take into account the shifting cosets.

Definition 3. The triple $\mathcal{F} = (G, \varphi, C)$ is called a *semi-frame* if all of the frame conditions hold, except that frame condition (iv) is replaced by the following condition:

(iv) $\hat{\varphi}_{xy} \mid \hat{\varphi}_{yz} = \tau \mid \hat{\varphi}_{xz}$,

where τ is the inner automorphism of $G_x/(H_{xy} \circ H_{xz})$ induced by C_{xyz}.

(In other words,

$$\tau(H_\xi) = C_{xyz}^{-1} \circ H_\xi \circ C_{xyz}$$

for every coset H_ξ of $H_{xy} \circ H_{xz}$.)

If $\mathcal{F}$ is a semi-frame, then the set A contains the identity relation and is closed under the operation $\otimes$ and the operation of conversion. The resulting Boolean algebra with additional operations has the same similarity type as a relation algebra. We shall denote it by $\mathfrak{C}[\mathcal{F}]$. (Actually, it is clear from the definition of $\otimes$ that A is automatically closed under this operation. No special hypotheses are needed, since A is closed under arbitrary unions. The real purpose of semi-frame conditions (iii) and (iv) lies in the proof of the next two theorems.)

In general it cannot be expected that all of the relation algebraic axioms will be valid in $\mathfrak{C}[\mathcal{F}]$. The Boolean axioms are of course valid, as are the distributive laws for $\otimes$ and conversion over Boolean addition, and the first involution law. The remaining axioms, the associative law, the identity law, the second involution law, and the cycle law, may all fail. The next theorem uses equations between cosets to characterize when each of these laws is valid in $\mathfrak{C}[\mathcal{F}]$.

THEOREM 8. *Suppose that $\mathcal{F} = (G, \varphi, C)$ is a semi-frame.*

(i) $R_{xy,\alpha} \otimes R_{yz,\beta} = R_{xy,\alpha} \mid R_{yz,\beta}$ *if and only if*

$$C_{xyz} = H_{xy} \circ H_{xz}.$$

(ii) *The identity law holds in $\mathfrak{C}[\mathcal{F}]$ if and only if*

$$C_{xxy} = H_{xy}$$

for all (x, x, y) in $\mathcal{E}_3$.

(iii) *The second involution law holds in $\mathfrak{C}[\mathcal{F}]$ if and only if*

$$\varphi_{xz}[C_{xyz}] = C_{zyx}^{-1}$$

for all (x, y, z) in $\mathcal{E}_3$.

(iv) *The cycle law holds in $\mathfrak{C}[\mathcal{F}]$ if and only if*

$$\varphi_{xy}[C_{xyz}] = C_{yxz}^{-1}$$

for all (x, y, z) in $\mathcal{E}_3$.

(v) *The associative law for $\otimes$ holds in $\mathfrak{C}[\mathcal{F}]$ if and only if*

$$C_{xyz} \circ C_{xzw} = \varphi_{xy}^{-1}[C_{yzw} \circ H_{yx}] \circ C_{xyw}$$

for all (x, y, z) and (x, z, w) in $\mathcal{E}_3$.

The theorem motivates the following definition.

DEFINITION 4. Let $\mathcal{F} = (G, \varphi, C)$ be a semi-frame. The following four conditions for all appropriate triples in $\mathcal{E}_3$ are called the *coset conditions*:

(i) $C_{xxy} = H_{xy}$,
(ii) $\varphi_{xz}[C_{xyz}] = C_{zyx}^{-1}$,
(iii) $\varphi_{xy}[C_{xyz}] = C_{yxz}^{-1}$,
(iv) $C_{xyz} \circ C_{xzw} = \varphi_{xy}^{-1}[C_{yzw} \circ H_{yx}] \circ C_{xyw}$.

From the theorem, one readily derives the following conclusion.

COROLLARY 3. *Suppose $\mathcal{F}$ is a semi-frame that satisfies the coset conditions. Then $\mathfrak{C}[\mathcal{F}]$ is a complete, atomic, measurable relation algebra.*

When $\mathcal{F}$ is a semi-frame satisfying the coset conditions, we shall call $\mathfrak{C}[\mathcal{F}]$ the (full) *coset relation algebra* on $\mathcal{F}$. The special case of this construction when we always take

$$C_{xyz} = H_{xy} \circ H_{xz}$$

gives just the group relation algebras, by part (i) of Theorem 8. Thus, every group relation algebra is a coset relation algebra. When

$$C_{xyz} \neq H_{xy} \circ H_{xz}$$

we can get algebras that are not group relation algebras, and hence not representable (by Theorem 4). For instance, the non-representable, measurable relation algebra $\mathfrak{A}$ constructed in Theorem 7 is an example of a coset relation algebra that is not a group relation algebra. Each of its cosets C_{xyz} is equal to $H_{xy} \circ H_{xz}$ except when $\{x, y, z\} = \{p, q, r\}$. In this last case, C_{xyz} is the unique coset that is different from $H_{xy} \circ H_{xz}$. (Recall that the quotient group $G_y/(H_{xy} \circ H_{xz})$ has exactly two cosets.)

Because coset relation algebras form a broader class of examples of atomic, measurable relation algebras than do group relation algebras, it is natural to reformulate Problem 1 as follows.

PROBLEM 2. *Is every atomic, measurable relation algebra essentially isomorphic to a coset relation algebra?*

The representation theorem for measurable relation algebras. We are finally in a position to formulate the principal theorem, a positive answer to Problem 2.

THEOREM 9 (Representation Theorem). *Every atomic, measurable relation algebra is essentially isomorphic to a coset relation algebra.*

The proof proceeds exactly as the proof of Theorem 3, except that it suffices to build a semi-scaffold $\langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ instead of a scaffold. This is quite easy to do. In order to construct the coset C_{xyz}, one selects a coset $H_{xz,\gamma}$ of H_{xz} such that

$$H_{xz,\gamma} ; a_{xz} \leq a_{xy} ; a_{yz},$$

and then puts

$$C_{xyz} = H_{xy} ; H_{xz,\gamma}.$$

(The right-hand side of the equation denotes the complex product of the two cosets—which are sets of functional elements—in the measurable relation algebra.)

Coset relation algebras are quite interesting and worthy of study in their own right. Aside from their role in the representation theorem, what other important properties do these algebras possess? To formulate one such property, let's denote by CRA the class of all coset relation algebras, their subalgebras, and isomorphic copies thereof.

THEOREM 10. CRA *is axiomatizable by a set of equations.*

The proof of the theorem is similar to Tarski's proof that the class of representable relation algebras is equationally axiomatizable (see [18]). However, it uses Theorem 9 instead of the Jónsson-Tarski representation theorem for atomic relation algebras with functional atoms.

It follows from Theorem 10 that the homomorphic image of any coset relation algebra, or even of any subalgebra of a coset relation algebra, must again be a subalgebra of a coset relation algebra. This interesting algebraic property is not at all evident from the definition of the algebras.

The notion of a coset relation algebra is really a generalization of the notion of a set relation algebra. It comprehends all set relation algebras (since it comprehends all group relation algebras), and it includes non-representable algebras as well. It is obtained by replacing points of a set relation algebra with groups and then allowing the natural operation of relational composition to be shifted by a coset multiplication.

Set relation algebras are usually considered to be the "concrete" or "standard" models of the theory of relation algebras. It seems reasonable, however, to propose that coset relation algebras be viewed as the standard models of the theory. They are certainly as concrete as set relation algebras, and they provide a more general class of standard models. This, of course, raises the question whether every relation algebra can be embedded into a coset relation algebra.

PROBLEM 3. *Does* CRA *coincide with the class of all relation algebras*?

A positive answer would provide a strong vindication of Tarski's axiomatization of the theory of relation algebras. On the other hand, a negative answer would lead to many interesting new problems. For example, is CRA finitely axiomatizable? Is the class of all representable relation algebras finitely axiomatizable over CRA? Is there a recursive algorithm for determining whether or not a finite algebra is in CRA?

It seems appropriate to conclude by recalling the final sentences from Tarski's original paper [15]:

> I do believe that the calculus of relations deserves much more attention than it receives. For, aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it.

Virtually everyone who has spent time working with relation algebras has learned to share Tarski's judgement.

Added in proof: The authors have recently shown that the answer to Problem 3 is negative. In fact, the class CRA is not finitely axiomatizable over the class of all relation algebras.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
MILLS COLLEGE
5000 MACARTHUR BOULEVARD
OAKLAND, CA 94613 USA
E-mail: givant@mills.edu

RÉNYI INSTITUTE OF MATHEMATICS
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST P. O. BOX 127, H-1364 HUNGARY
E-mail: andreka@renyi.hu