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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Geoffrey Hellman, Thomas Jech, Wolfram Pohlers, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

BRUNO POIZAT. ***A course in model theory. An introduction to contemporary mathematical logic.*** English translation by Moses Klein of JSL LVIII 1074. Universitext. Springer, New York, Berlin, Heidelberg, etc., 2000, xxxi + 443 pp.

The present work is the English translation of a text which was privately published in French in 1985, and was reviewed in ***The journal of symbolic logic*** by Palyutin (JSL LVIII 1074). That review is equally applicable in its entirety to the present translation, apart from its last sentence: "The book is very useful and it is a pity that there is no English translation yet. . . . "

The text provides an introduction to model theory which manages to go fairly deeply into key concepts of stability theory. In this respect it is somewhat comparable to the text by Wilfrid Hodges, ***A shorter model theory*** (Cambridge University Press, 1997), and is certainly suitable, among other things, for an introductory graduate course in model theory. Though now over fifteen years old, Poizat's text has aged well. For a detailed description of its contents, its virtues, and its idiosyncrasies, we may refer the reader to the review by Palyutin. The reviewer bought the third copy sold of the French edition, and would still recommend it to anyone interested in model theory.

The author begins with seven chapters of basic logic and model theory (compactness, arithmetic) and one on basic set theory of a purely utilitarian character. After this preparation, the remaining twelve chapters of the text consist of a course on contemporary "pure" model theory (variously known today as "stability theory" or "classification theory"). This course begins, in the ninth chapter, with the words: "We are now embarking on what can

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properly be called model theory" (p. 181). There follow eleven chapters of general theory in a classic presentation which includes such matters as the elimination of imaginary elements and the relationship of the general theory to the Galois theory of differential equations. The book concludes in Chapter 20 with an application of this general theory to Shelah's "main gap" problem, which is concerned with estimating the number of models of an arbitrary theory in a fixed cardinality. Enough is done to see the theory at work on a very broad problem; the messier technical aspects are summarized very briefly, but cogently, in the final two sections.

Conspicuous by its relative absence is the geometric stability theory which has since enriched the subject considerably, and is heavily implicated in Hrushovski's applications of this subject to number theory. The geometrical theory has its roots in work of Zilber beginning in the early seventies, mentioned in a historical note on page 350, which became thoroughly integrated with the general theory at a later stage. For more on this topic, after mastering the present text or an equivalent, one may now consult Anand Pillay's excellent ***Geometric stability theory*** (JSL LXIII 1190). One wonders where Poizat would have left his subject had he undertaken the same project today—perhaps at much the same point, though at the cost of a greater measure of self-restraint.

As far as the translation is concerned, it is relatively faithful both in style and content, and has the virtue of not overly concealing the text from which it is derived. As much of the humor and elegance of the original survive as could reasonably be desired. The translator is a bit cavalier about both tenses and genders, which have after all been chosen by the author. On the other hand one may wish he had converted Poizat's "dope," less implausible in French, back into DOP, since this is a key and standard technical term. Various slips occur, for the most part affecting style more than content. Occasionally proofs are a bit hard to follow because of slight inaccuracies in the translation; on the other hand an occasional misprint in the original has been silently corrected. The French version is unusually clear; the English translation is as clear as most recent texts written originally in English, with occasional rough spots. Without belaboring the point, a few examples relating both to style and content may be in order. In the last sentence of the introduction, replace "I dare hope only" by "I dare not hope"; on the previous page, replace "ignored by" by "unfamiliar to"; in the proof of Theorem 12.18, on page 252, mid-page, read "at a point beyond which" rather than "from which."

It is surprising that in a book from this publisher the epigraphs, which are given in the original French, are not always given accurately. In particular the substitution of *minime* for *mimine* produces a noteworthy effect.

While the French original is on occasion significantly clearer than its translation, the present edition of a classic text is very welcome and valuable. Copies of the French original remain available, and may be obtained from the author (Institut G. Desargues, Université Claude Bernard, Lyon I, Villeurbanne Cedex 69622, France).

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Sy D. Friedman. ***Fine structure and class forcing.*** De Gruyter series in logic and its applications, no. 3. Walter de Gruyter, Berlin and New York 2000, x + 221 pp.

Sy Friedman's book brings together an impressive body of his work over the past twenty years. It has been said that this work lies in two areas—in fine structure and in class forcing. But all of his deep results regarding class forcing rely on the fine structure of L. The two are, in fact, inseparable.

The book is organized around what might be called the generalized Martin's axiom content of "$0^{\#}$ exists" for L-amenable class forcings. Friedman is interested in L-amenable class forcings that have $L[0^{\#}]$-definable generic filters. In this way, he can treat class genericity—otherwise a second-order notion—in the first-order theory ZFC + "$0^{\#}$ exists."

Chapter 1 develops fine structure for L using Jensen's "Σ^* theory." As Friedman points out in his *Preface*, the reader primarily interested in class forcing can skip the first chapter. The more sophisticated examples of class forcing are, however, inaccessible without it. Fine structure plays a key role in Jensen coding, especially in the presence of $0^{\#}$.

A basic idea of Jensen's going back to his original work on the fine structure of L is to reduce Σ_{n+1} definability to Σ_1 definability relative to predicates for projected Σ_n satisfaction. He uses this to define Σ_n-Skolem functions for the J-hierarchy. Over rudimentarily closed transitive structures, Σ_1-Skolem functions are uniformly Σ_1-definable. But as anyone familiar with Devlin's books—for example ***Constructibility*** (JSL LII 864)—will recall, in general Σ_n-Skolem functions are Σ_n-definable, but not uniformly so. This is an abiding complication. Parameters that are not uniformly Σ_n-definable figure in definitions and substantial care is required when working with transitive collapses. The Σ^* theory eliminates this complication by working relative to a special hierarchy of Σ^*_n formulas, rather than the usual Lévy hierarchy. Friedman develops the Σ^* theory from scratch and uses it to prove a global $\Box$ principle as well as to construct a global gap-1 morass with attached $\Box$. These, relativized to "reshaping sequences" (in the sense of Beller–Jensen–Welch, JSL L 1081), are needed for Jensen coding, as well as some of the difficult class-forcing results that appear later in Friedman's book. The first chapter ends with a construction of a gap-2 morass.

Chapter 2 begins textbook-style with a standard development of set forcing using Shoenfield forcing terms. The second section of Chapter 2 introduces class forcing. In this book Friedman is primarily interested in class forcing over inner models close to L. Nevertheless, he gives general definitions. He requires ground models to be models of ZF of the form $\langle L(A), A\rangle$, with the class partial ordering P definable over this structure. This is so that the ground model and P are uniformly definable in outer models by using a predicate symbol for A. He defines a filter G on P to be *generic* when it meets every ground model definable dense subclass of P. Under this definition, generic extensions are generic with respect to a language containing predicate symbols for ground model definable classes of forcing terms, as well as names for elements of the extension. This is a natural choice because he is largely interested in L-amenable class forcings for which a generic class is definable over $L[0^{\#}]$. In this setting, those classes are definable.

"Tameness" is a combination of two features that insure a class forcing is well behaved and preserves ZFC. Friedman calls a forcing *pretame* if it has the first of these properties, namely that, given a uniformly definable set-indexed family of dense classes $\langle D_i : i \in I\rangle$ and a condition p, there exist sets $d_i \subseteq D_i$ and a condition q extending p such that the d_i's are predense with respect to q. Pretameness implies that the forcing relation is definable over the ground model. (Unlike set forcing, this can fail in the case of class forcing.) Furthermore, pretameness implies that collection and separation are preserved in generic extensions, even in a language with predicate symbols for ground model definable classes of Shoenfield terms. Conversely, if generic extensions via P preserve collection and separation in this expanded language, then P is pretame. The second part of tameness similarly characterizes preservation of the power-set axiom.

Chapter 3 introduces what might be considered the organizing topic of the book, namely, the construction from $0^{\#}$ of generic classes for L-amenable class forcings. The first section is a concise modern development of the theory of $0^{\#}$. A sketch of Jensen's covering theorem and short proofs of Carlson's and Magidor's strong covering theorems are included. Friedman defines an L-amenable class forcing to be *relevant* if a generic filter for it is definable, perhaps with parameters, over $L[0^{\#}]$ and ends the chapter with several examples of factorable relevant forcings.

Chapter 4, which proves Jensen's coding theorem, enters substantially more technical territory. This is where some of the fine structure developed in Chapter 1 comes into play. Friedman first handles the case in which $0^{\#}$ does not lie in the universe to be coded by using "coding delays," a device of his that simplifies extending conditions. A complicating feature of Jensen coding is that distributivity and extendibility proofs are interdependent. In the third section of Chapter 4, Friedman gives his simplified proof of Jensen's coding theorem in the general case. Roughly, he uses coding based on Jensen's limit coding, but thinned using $\square$ and $\diamondsuit$ in the way a κ^+-Souslin tree thins κ^+-Cohen forcing.

Chapter 5 contains several interesting applications of Jensen coding. In the first section, Friedman proves the existence of a real that is not generic for any L-amenable tame class forcing and that has constructibility degree strictly less than $0^{\#}$. Roughly speaking, this real is generic for a forcing that is definable in L when constructed (by using $0^{\#}$) out to the next L-cardinal beyond the end of the universe, but diagonalizes out of all the L-amenable tame class forcings, which are constructed before that point.

In the second section, he gives a nice proof of the Paris gap theorem from covering and a proof of Solovay's well-known but unpublished result that the Paris gap theorem is best possible. These are valuable additions to the published literature.

He goes on to consider "strict genericity," showing that the following "antifactoring" phenomenon, which cannot occur in set forcing, does occur in class forcing: There exists a real R of constructibility degree strictly less than $0^{\#}$ such that R lies in a generic extension of L, but $L[R] \neq L[G]$, for any G that is generic for an L-definable tame forcing.

Finally, in this section Friedman proves another remarkable theorem, namely, that over a minimal model of ZFC + "$0^{\#}$ exists" it is possible to construct a generic class of ordinals A such that $0^{\#} \notin L[A]$, yet $L[A]$ has no proper elementary substructures.

The third section of Chapter 5 proves Friedman's generic saturation theorem—if the ordinals are $\omega + \omega$-Erdős and if P is an L-amenable forcing for which there exists a generic class definable over V, then there exists a generic filter on P that is definable over a set-generic extension of $L[0^{\#}]$. Very roughly, the idea is to use the Erdős property and a (set-forcing) collapse to get a countable fragment of a generic filter that can be stretched to a full generic filter by using the indiscernibles provided by $0^{\#}$.

Chapter 6 deals with Friedman's work on Π^1_2 singletons. It gives his proof that there exists a class-generic Π^1_2 singleton of constructibility degree intermediate between 0 and $0^{\#}$. In the presence of $0^{\#}$, the Π^1_2 formula that this real satisfies can have at most one solution, but this is not a theorem of ZFC. He shows that this result can be improved to get a Π^1_2 singleton of intermediate constructibility degree that is provably a singleton in the theory ZFC + "for each n, there exists an n-ineffable cardinal." The chapter ends with several applications of descriptive set theoretic flavor.

Chapter 7 addresses which classes of ordinals are "admissibility spectra" of reals constructible from $0^{\#}$, that is, which classes of ordinals are of the form $\Lambda(R) = \{\alpha : L_\alpha[R]$ is admissible$\}$, for some real R of constructibility degree strictly less than $0^{\#}$. Using a relatively easy extension of the basic coding proof, Friedman shows that, given any class X of ordinals that is Σ_1-definable (without parameters) over L and contains all L-cardinals, there exists a real $R <_c 0^{\#}$ such that $\Lambda(R) \subseteq X$. It is much harder to arrange that $\Lambda(R) = X$. For this, care must be taken to preserve the R-admissibility of ordinals inside X, as well as to destroy the R-admissibility of ordinals outside X. This requires an extension of the coding method that is an order of magnitude more delicate, namely, Friedman's method of "strong coding." Using this method he shows that there exists a real strictly below $0^{\#}$ with admissibility spectrum equal to the class of recursively inaccessible ordinals (admissible limits of admissible ordinals). Friedman observes that this result generalizes in a rather technical way and proves a number of limitations on results of this sort. Finally, he proves an "anticharacterization" result, showing that there cannot be any simple characterization of admissibility spectra that are realized by reals constructible from $0^{\#}$.

Chapter 8 contains an assortment of four topics falling under the rubric *applications*. The first of these is Δ_1-coding. In Jensen coding, a coded class of ordinals can be recovered from a coding real by recursion on cardinals, hence is Δ_2-definable from the real. Friedman provides a sufficient condition for classes to be Δ_1-codable. He also proves an anticharacterization result, showing that there cannot be a simple characterization of L-amenable classes that are Δ_1-definable from reals strictly below $0^{\#}$. Next, he develops apparatus that suffices to iterate many important examples of class forcing in a special manner. He uses this to iterate Jensen coding to get that his generic saturation theorem is sharp with regards to indiscernible preservation. Minimal coding is Friedman's third application. He gives a proof that it is possible to produce a real coding the universe that is of minimal constructibility degree. The proof is difficult and requires revisiting fine structure theory. In the book's final section, Friedman proves a number of interesting results regarding regularity properties of projective sets of reals.

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Neil Immerman. ***Descriptive complexity.*** Graduate texts in computer science. Springer, New York, Berlin, and Heidelberg, 1999, xvi + 268 pp.

Overview. In a slim and neatly packaged volume of 250 pages, Neil Immerman has written a definitive reference on descriptive complexity, the field that attempts to elucidate the study of resource-bounded computation by describing the semantic complexity of an algorithm by its syntactic formulation in logic. As such, it is a specialty within (and I might add the primary motivating force behind) the burgeoning area of definability over finite structures known as finite model theory. What makes this text different from other references is its emphasis on logical definability over classes of structures whose vocabulary is assumed to include not only a linear ordering but also arithmetic on the domain.

Background. What are the main ideas behind this logical view of computation? Instead of quantifying asymptotic physical resources such as time and space required to execute an algorithm on a given model of computation, logic provides a machine-independent approach to the inherent asymptotic difficulty of a problem. Problems are naturally expressed as mappings from finite input structures to output structures—otherwise known as queries on databases. Tractable queries (those computable in polynomial time or logarithmic space, for example) can be expressed by using formulas drawn from extensions or restrictions of first-order or second-order logic. Because the focus is on characterizing asymptotic resources for finite computations, these logical descriptions lead naturally to the study of definability over finite structures: finite model theory.

Using logical formulas to describe computations has a long history. Consider the class of *binary string* structures given by a finite successor chain with a unary predicate. Call a set R of binary strings *regular* if membership in the corresponding class of structures R' can be recognized by a finite state automaton (a type of computer that changes state as it moves along the chain). An old (1960) theorem of Büchi says that R is regular if and only if R' is definable in monadic second-order logic (JSL XXVIII 100). One of the best places to study these results is a nice book by Howard Straubing, ***Finite automata, formal logic, and circuit complexity*** (Birkhäuser, 1994). Broader connections between logical definability and computational complexity were realized later, when in 1974, Fagin showed that definability in full existential second-order logic captured non-deterministic polynomial time bounded computation (**NP**) over unordered structures. The next decade brought refinements of the technique to yield tantalizing results on ordered structures that captured **P** (polynomial time) via inductive definability (LFP for least-fixed-point logic) and **L** (logarithmic space) via restricted recursion using path quantifiers (TC for transitive-closure logic). Even a correspondence with elementary definability in first-order logic was achieved, as those computations that could be computed in a constant amount of time on the CRAM, a concurrent random access machine (a powerful parallel model of computation in which multiple processors have simultaneous non-conflicting access to a global memory). (An alternative characterization was given in my paper *A constant-space sequential model of computation for first-order logic*, ***Information and computation***, vol. 143 (1998), pp. 231–250.) These last results, however, hold only with the additional numerical predicate *bit* added to the vocabulary (which is equivalent to having $+$ and $\times$).

Format. These descriptive classes are all explained in great detail with many examples and exercises throughout the book. Proofs of theorems are complete and are kept as simple as possible. Running commentary through most proofs makes it possible to understand the gist of an argument without getting bogged down in technicalities if one so desires. Supplementary figures are drawn with great care and detail, and are a welcome aide to many arguments.

Contents. After two background chapters in logic and complexity respectively, the core material of the book commences with first-order reductions (elementary definability turns out to be such a weak class from a complexity-theoretic viewpoint that its primary use appears to be in relating queries of higher definitional complexity). These reductions are used to display complete problems for the classes **L**, **NL**, and **P**. From this base, it is natural to consider inductive definitions, which are shown to express exactly those queries in **P** over ordered structures. A finer analysis, based on the closure ordinal, reveals a tight (linear) correspondence between the depth of an induction and the parallel time it takes to evaluate it on the CRAM. Further topics in this section explore the correspondence between the number of variables and the number of processors required, and connections with other forms of parallel computation such as Boolean circuits and alternating models. Another important covered topic is the locality of first-order queries and their linear-time complexity on structures of bounded degree.

The book then turns to more advanced material, starting with second-order definability. Here the reader will find a detailed proof of the correspondence between **NP** and existential second-order logic, along with coverage of **NP**-completeness and the polynomial hierarchy (= full second-order logic). Restricting attention to monadic second-order yields lower bounds (via games) for graph connectivity. Attention then turns to a finer analysis of fixed-point (inductive) definitions. Notable among the results is the author's own that fixed-point is closed under negation on classes of finite structures (one of the few pleasant places where finitary definability is better behaved than its infinitary counterpart). A similar normal form for transitive-closure logic—a natural restriction of fixed-point logic to recursion along paths—holds only over ordered finite structures. Following on this is the author's celebrated result that non-deterministic space is closed under complement.

The remaining chapters show increased specialization, with material on the use of non-monotone inductions (partial fixed-point logic) to characterize polynomial space, and the connection between Boolean circuits and built-in relations that depend only upon the size of the domain. The most important of these relations is a total ordering ($<$), whose role in logics describing feasible computations such as polynomial time appears indispensable, even though the resulting query is independent of that order. Unfortunately, this has the defect of allowing definitions with no effective way of deciding whether they might yield order-dependent non-queries (which break isomorphism types over the original unordered structures). The big question then becomes: *Is there a recursive enumeration of the polynomial time queries?* A positive resolution would imply a database programming language that expresses exactly all feasible queries. Subsequent examination of graph coloring and logics that can count provides some insight into this fundamental problem, but any answer remains elusive.

An equally important task would be the use of finite model theory techniques to separate complexity classes. Progress in this direction is illustrated by the use of arguments attempting to show that certain problems are not first-order expressible (with and without counting). Current techniques, however, do not work over vocabularies that are strong enough to describe conventional machine-based computations. It is unfortunate that we do not have any techniques from finite model theory that imply separations of complexity theoretic significance in this setting.

The book concludes with applications to database theory and computer-aided verification, along with the author's personal views on the significance of descriptive complexity to the larger picture of computational complexity (such as **P** versus **NP**) and computational correctness (software reliability). All in all, this book brings a perspective of the exciting area of complexity theory within the reach of the logician. Neil Immerman has done a tremendous service to future researchers, and I for one can acknowledge that his contribution will be greatly appreciated.

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GREG RESTALL. ***An introduction to substructural logics.*** Routledge, London and New York 2000, xiv + 381 pp.

"Substructural logics," the designation for logics with restricted Gentzen structural rules, is now ten years old. During these ten years, authors writing about substructural logics and using the designation felt obliged to explain apologetically what these logics are. Maybe times are coming when they will not have to do so any more. A sign of these times seems to be that in the Mathematics Subject Classification for the year 2000 there are now two numbers for substructural logics: 03B47, Substructural logics (including relevance, entailment, linear logic, Lambek calculus, BCK and BCI logics); 03F52, Linear logic and other substructural logics. Another sign may be the appearance in the same year of the first textbook (although not the first book) with "substructural logics" in the title—namely, the book reviewed here. To give an idea of what is in this book, and of what is not in it, I will first say something about substructural logics in general. I will mention the book later, at appropriate places.

The designation "substructural logics" and a unified approach to the subject matter are still relatively new, but the main substructural logics themselves are fairly old today. Relevant logic, the logic that rejects thinning, has its earliest roots in the 1920's, in the milieu that produced the axiomatization of intuitionistic logic, and it was also anticipated in the 1930's in the λI-calculus. This logic grew steadily in the second half of the twentieth century, and the literature on it overwhelms by sheer volume the literature on all the other substructural logics. Nevertheless, in broader logic circles it was not quite in the main stream. The Lambek calculus, which rejects thinning, contraction, and permutation, dates from the late 1950's. It acquired some notoriety in the 1980's, in particular among linguists and logicians close to theoretical linguistics. Affine logic, which rejects contraction, and which used to be known as BCK logic, has roots going back to the 1930's. It was developed to a greater extent in the 1970's, but did not receive much attention before the advent of linear logic.

Linear logic, which rejects contraction and thinning, stems from work done in category theory in the 1960's, when the important notion of symmetric monoidal closed category was formulated, and when Gentzen's techniques were applied in studying it. The name "linear logic" is justified by the connection with linear algebra, which is made via symmetric monoidal closed categories. There were other anticipations, but it was in the late 1980's that linear logic came to the fore, and achieved notoriety unmatched by any other substructural logic. In the first half of the 1990's it was very fashionable.

From a philosophical point of view, relevant logic may seem the most interesting among substructural logics, but, from a mathematical point of view, linear logic is central. This logic is at the branching point in the hierarchy of the main substructural logics: relevant and affine logic extend linear logic, and the Lambek calculus is included in it. Besides that, the connection of linear logic with important mathematical structures is the clearest. The simultaneous rejection of contraction and thinning has a mathematical rationale: these principles go together (they come from the unit and counit of the adjunction of the product bifunctor with the diagonal functor).

The success of linear logic, however, cannot be explained only by its mathematical importance. This success is also a phenomenon that reflects the present condition of logic. Linear logic became popular on the border of logic with theoretical computer science, at a time when there was considerable movement across this border. It was presented deftly, as a multifaceted new product, even with a new notation. And it drew much from the authority its chief proponent, Jean-Yves Girard, has in proof theory.

Linear logicians use the label "substructural" very seldom, if ever. If they need such a label, they would probably prefer the rival labels "resource-conscious" or "resource-sensitive," which have overtones pointing to extralogical spheres (not only the sphere of computing—one cannot help thinking about the sphere of economics, too). The label "substructural" is encountered rather in non-linear substructural logics. The author of the book reviewed here comes from the Australian school of relevant logic.

The project to unify the field of substructural logics could be perceived as an attempt by non-linear substructural logicians to associate themselves with the success of linear logic. Their pretense of drawing dividends from this success is, however, legitimate. Many features of linear logic that were highlighted ten years ago as novelties were in fact well known among other substructural logicians, and are not peculiar to linear logic.

Much was made, for instance, of the distinction between the *multiplicative* and *additive* (i.e., *lattice*) connectives, which is in fact an old distinction, well known, but not under this name, in relevant and affine logic.

Next, models in the style of Kripke, similar to the phase models of linear logic, were investigated intensely in relevant logic starting from the early 1970's. In particular, a large amount of work on these models was done by the Australian school. Greg Restall concentrated mainly on that topic, to which at least one-third of his book is devoted. The book is stronger on that side than on the side of proof theory. It provides a rather detailed guide to representations of distributive and non-distributive lattice-ordered residuated groupoids in structures analogous to Kripke models.

A peculiarity of linear logic is usually taken to be the modality of the S4 type to which one adds modalized versions of the missing structural rules. Such modalities were anticipated in relevant logic (see L. F. Goble, *A system of modality*, ***Notre Dame journal of formal logic***, vol. 12 (1971), pp. 225–237, where a modality of the S4 type with a modalized version of thinning is added to the relevant logic R), but not much was done with them, indeed. It is not forbidden, however, to add modalities of this kind to any substructural logic, and obtain, as easily, the same kind of modal translations usually associated with linear logic. Restall's book does not dwell much on modal translations, which are considered an important topic in linear logic. In the chapter on modalities it concentrates rather on other matters (for example, on Wansing's treatment of modalities in the style of Belnap's display logic).

Restall does not deal in this book with predicate logic, and so he does not mention Grishin's important translation of classical predicate logic into the decidable affine predicate logic. Grishin, who has much merit for having thoroughly investigated affine logic in the 1970's, is not mentioned at all in the book. This is a notable lacuna in the bibliography, which is otherwise quite extensive. (Another lacuna is the omission of Jaśkowski, who should be mentioned not only for his investigation of linear and affine implication from the early 1960's, but should also be remembered for his discovery of natural deduction, contemporaneous with Gentzen's.)

Another big novelty of linear logic is usually taken to be the proof nets, which are officially presented as an alternative syntax, but which might be taken as a kind of geometric model for deductions. Proof nets for linear logic are based on coherence for symmetric monoidal closed categories (see G. M. Kelly and S. Mac Lane, *Coherence in closed categories*, ***Journal of pure and applied algebra***, vol. 1 (1971), pp. 97–140, 219). This coherence is best understood as a faithful modeling of free symmetric monoidal closed categories in a certain category of tangles, in the sense of knot theory (for the categorical treatment of tangles see C. Kassel, ***Quantum groups***, Springer-Verlag, 1995, Chapter XII, and references therein). Coherence for symmetric monoidal closed categories explains why proof nets work in linear logic, and it is easier to grasp this coherence than the mechanism of proof nets. We do not know of such a coherence for *closed* categories corresponding to relevant and affine logic, and we do not have proof nets for these logics. Restall regrets in the introduction he could not cover proof nets because of lack of space, and because it is unclear how to extend them to other substructural logics. In a chapter devoted to categories, he comes to the verge of defining the general notion of symmetric monoidal closed category, but at that point he prefers to switch to Girard's particular symmetric monoidal closed category of coherence spaces (the "coherence" of these spaces should not be confused with the coherence mentioned above).

I do not find that chapter on categories very useful. At the beginning of the chapter, the reader should have been told that in categories we may have two different arrows with the same source and the same target, i.e., two different deductions with the same premiss and the same conclusion. This is an essential matter. It marks the watershed between proof theory and the rest of logic, where one deals with consequence *relations*, i.e., categories that are preorders. There are some inaccuracies in this chapter. On page 212, the definition of small category is not standard: the class of arrows should also be a set. On page 221, the definition of cartesian category is not standard. A category is usually called *cartesian* if it has all finite products, including the empty product, i.e., the terminal object. It is often called *bicartesian* if it has all finite products and all finite sums. It is not called cartesian if it has non-empty finite products and non-empty finite sums. Coherence for monoidal and symmetric monoidal categories was not proved by Mints and Lambek, as claimed on page 227, but by Saunders Mac Lane (*Natural associativity and commutativity*, ***Rice University studies***, *Papers in mathematics*, vol. 49 no. 4 (1963), pp. 28–46). Altogether, there are in categories many more strings to the substructural bow than one can gather from this chapter.

Restall devotes two chapters near the end of the book to decidability and undecidability. He presents Urquhart's undecidability results for distributive relevant propositional logics and Lincoln, Mitchell, Scedrov, and Shankar's undecidability result for linear propositional logic with the modalities. The result that affine propositional logic with the modalities is decidable is not mentioned (see A. P. Kopylov, *Decidability of linear affine logic*, ***Proceedings, Tenth annual IEEE symposium on Logic in Computer Science***, IEEE Computer Society Press, 1995, pp. 496–504; I know this paper only from references).

Restall's book stays at the level of propositional logic, which is probably not a serious shortcoming, because that is where most of the interesting things happened in substructural logics up to now. There are, however, remarkable facts about substructural predicate logics, too. Speaking of decidability, a textbook would be a good place to present a complete proof based on cut elimination of the folkloric result that linear predicate logic without the modalities is decidable.

I will make now a few critical remarks on the introductory sections of this textbook. In the introduction, the book is said to presuppose a first course in logic, but then at the beginning it moves too slowly. The definition of languages on page 14 ff is excessively formalistic, and makes opaque very elementary matters, which the readers presumably already know, after their first course. The readers might imagine that the adjective "substructural" is somehow derived from the *substructures* of page 19. Conjunction and disjunction on page 35 turn into greatest lower bound and least upper bound after "Lindenbaumization"—they are not that at the outset. The notion of structural rule should cover rules that we can formulate without mentioning any constant of the language to which the premisses and the conclusion belong, rules that apply independently of what constants we have in the language. I am not convinced that Restall's notion of a structural rule on page 24 is that notion. Restall should be complimented, however, for not following the bad habit of many authors who call rules for the logical constants, i.e., Gentzen's rules that are not structural, simply "logical rules." Structural rules are also logical—even more so.

The notion of structural rule is of particular importance in the field of substructural logics. As with many other things, we owe this fundamental notion to Gentzen (but I have learned from Peter Schroeder-Heister that Frege had a premonition of sequents and of that notion, in particular in §§47–48 of the ***Grundgesetze***; cf. P. Schroeder-Heister, *Frege and the resolution calculus*, ***History and philosophy of logic***, vol. 18 (1997), pp. 95–108). The importance of structural rules is corroborated by their conforming to a fundamental pattern one finds not only at the level of propositional logic, but also at the level of predicate logic, in the combinators underlying the binding of variables. This combinatory pattern is found all over mathematics. It starts with something monoidal, i.e., something that has multiplication with unit and associativity, it is followed by commutativity, and then it moves to projections and diagonal maps.

The critical remarks made in the course of this review were about particular points, and I wanted to be done with these particulars before giving a general opinion. I suppose this textbook can be quite useful for beginning graduate students, and that more in departments of philosophy than in departments of mathematics. I understand this is the public addressed by the author. The style of the exposition is less suitable for educated logicians who would like to familiarize themselves with substructural logics, but they too, and even specialists, could profit considerably from some sections, mostly in the third and fourth part.

The first part of the text covers introductory matters and proof theory. There the reader can learn something about Gentzen's sequent systems, natural deduction, the Curry–Howard correspondence, and beta normalization. Inspired by Curry and by Belnap's display logic, the author formulates general conditions that guarantee cut elimination. The chapter on theories, in the first part, is closely tied to the third part of the book, which is about models in the style of Kripke, and so are both chapters on algebraic models in the second part, which contains also the chapter on categories. The fourth part is about decidability. The book ends with a brief chapter about certain points of a philosophical nature concerning models in the style of Kripke.

I have kept a commendatory statement for the very the end of the review (which is the place where publishers fish for blurbs). I cannot but welcome the appearance of this book.

(Errata: Page 37, line 8, Peirce's Law is $(p \supset q) \supset p \vdash p$. Page 42, line 6 from below (f.b.), early 1950's. Page 93, line 19, $w \supseteq y$. Page 224, lines 16–17, $R(X)$ should be $L(X)$, twice. Page 227, line 1 f.b., x *coheres* with y. Page 228, line 3, $a \in \mathcal{A}$, $\{x, y\} \subseteq a$. Page 230, lines 7 f.b. and 2 f.b., $y = y'$ (mod $\mathcal{B}$) is undefined. Page 232, line 9, *Introduction to Higher Order Categorical Logic*. Page 356, in [134], R. Jacobson, in [138], A. Scedrov.)

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G. Aldo Antonelli. *Proto-semantics for positive free logic*. ***Journal of philosophical logic***, vol. 29 (2000), pp. 277–294.

Negative free logic (including E+-logic, the logic of partial terms, and the logic of definedness) has had a widely accepted *standard* (model-theoretic) semantics since Rolf Schock first discovered it forty years ago (JSL XXXVII 423, 424). Antonelli's proto-semantics is intended to found a similarly plausible bivalent, standard semantics for positive free logic (including E-logic and the logic of existence). The quest for a standard semantics for positive free logic is driven by a number of applications. Analyzing the role of irreferential singular terms in natural language and in philosophical and mathematical discourses inspired the development of free logic. In the specification of computer systems, positive free logic is natural for formalizing non-strict programming language constructs, such as the **if-then-else** found in many programming languages. Plausible standard semantics motivate applications.

Developing a widely accepted standard semantics for positive free logic has been surprisingly difficult. About the same time as Schock's semantic discoveries for negative free logic, Belnap and Lambert had independently developed a model-theoretic semantics for positive free logic, now known as the *outer domain semantics*, but they and a number of others including Church (JSL XXX 103) found this semantics philosophically objectionable because it populates the outer domain with "non-existent objects." Semantics using the substitution interpretation of the quantifiers or probability functions would not do because they are *non-standard*. Insofar as, for example, truth-value semantics for positive free logic suggests a model-theoretic counterpart, it is an outer domain semantics, with only countable domains at that. Supervaluation semantics is model-theoretic enough, but it is problematic because, amongst other reasons, its valid sentences are not recursively enumerable.

For concreteness, Antonelli's proto-semantics is designed to fit Lambert's axiomatization (JSL XXX 103) of first-order languages with identity and definite descriptions, but no function symbols. There is a (possibly empty) *domain* (of existing objects), but no outer domain of non-existents. A *partial reference function* ρ maps the singular terms into the domain. A *proto-interpretation* assigns each predicate a *signed extension* of either the form $(S, +)$ or $(S, -)$, where S is a relation of the appropriate arity on the domain, and the sign guides the assignment of truth-values to atomic sentences containing non-denoting terms. (That is, '$+$' signals true, and '$-$' false.) An *interpretation* Π assigns each n-tuple of terms a proto-interpretation subject to two restrictions that insure extensionality. (The restrictions are reminiscent of the identity normality conditions in Leblanc's truth-value semantics, JSL XLIII 376.) Finally, a *model* is a pair of the form (Π, ρ). The clause for identity in the recursive definition of truth reads as follows:

$$(\Pi, \rho) \models t_1 \equiv t_2 \quad \text{iff} \quad \begin{cases} \rho(t_1) = \rho(t_2) & \text{if either } \rho(t_1) \text{ or } \rho(t_2) \text{ is defined} \\ \Pi(t_1) = \Pi(t_2) & \text{otherwise} \end{cases}$$

A minor problem with the semantics arises because identity is relative to the monadic predicates in a language. For example, if a language has exactly one monadic predicate P, then in proto-semantics we have

$$\models \neg E!(t_1) \wedge \neg E!(t_2) \wedge (P(t_1) \leftrightarrow P(t_2)) \rightarrow t_1 \equiv t_2$$

but this sentence is not provable in Lambert's axiomatization. Intuitively, a corresponding outer domain model can be read off from a proto-semantic model, and the outer domain's population is limited to at most 2^n distinct non-existents, where n is the number of monadic predicates in the language. When there are infinitely many monadic predicates, compactness is compromised. Indexing the signed extensions, much as is done in the semantics of some modal logics with "possible worlds" drawn from an arbitrary set, eliminates the technical problem, though possibly weakening the intuitive appeal of the semantics.

In addition to establishing completeness, Antonelli uses proto-semantics in providing new proofs of compactness, the Craig interpolation lemma, and (two versions of) the Beth definability theorem. In the proof of the Craig interpolation lemma, novel use is made of definite descriptions to supply an infinite batch of non-denoting terms. As usual, the Craig lemma is then used to establish the Beth theorem. The model-theoretic proof proceeds in much the same way as in standard first-order logic. The present paper carefully distinguishes weak and strong forms of the Beth theorem for positive free logic, whereas earlier proofs in the literature did not. The strong form of the theorem is useful because (deviating from Antonelli's terminology and conventions) when free variables can be non-denoting (JSL L 548), the validity of an open sentence implies the validity of its universal closure, but not conversely.

I find proto-semantics to be the most philosophically palatable standard semantics to date both fitting Lambert's axiomatization of positive free logic and meeting Church's demand that the semantics "characterize in [a] distinctive way a logic that allows true assertions having denotationless names as their subjects." Nevertheless, the simplicity of outer domain semantics insures that it will remain in the free logician's tool box as one of a set of "consistent and complementary semantics" for positive free logic. Perhaps outer domain semantics too can gain wider acceptance as a standard semantics for positive free logic in cases where the denizens of its outer domain are more perspicuously viewed as "irrelevants" rather than as "non-existents," nullifying Church's stricture. For example, in axiomatizing the natural numbers, *pred*(0) (the predecessor of 0) might be either a natural number or an alien from the outer domain. In other words, *pred*(0) has a denotation but it is irrelevant. The behavior of irrelevants can be left unspecified.

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KAI F. WEHMEIER. *Fragments of HA based on* Σ_1*-induction*. ***Archive for mathematical logic***, vol. 37 no. 1 (1997), pp. 37–49.

This article investigates the intuitionistic version $iI\Sigma_1$ of $I\Sigma_1$ (in the language of PRA). $I\Sigma_1$ stands for induction restricted to Σ_1-formulas. In the first part a soundness theorem for the notion of recursiveness and q-realizability is proved. From the soundness theorem some corollaries are derived. In particular it is shown that the provably recursive functions of $iI\Sigma_1$ are exactly the primitive recursive functions. Furthermore an unpublished theorem of A. Visser is sharpened: $iI\Sigma_1$ together with induction over arbitrary prenex formulas is Π_2-conservative over $iI\Pi_2$.

In the second part of the paper the relationship between $iI\Sigma_1$ and $iI\Pi_1$ (now in the usual language of arithmetic) is studied. By model-theoretic means it is shown that every provably recursive function of $iI\Pi_1$ can be majorized by a polynomial. Together with a result of Buss, it follows that $iI\Sigma_1$ and $iI\Pi_1$ are mutually incomparable. $iI\Pi_1$ also lacks closure under Markov's rule.

The article is quite understandable. Some results, especially Visser's theorem, are surprising. The underlying concepts and methods should be familiar.

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WOLFGANG BURR and VOLKER HARTUNG. *A characterization of the* Σ_1*-definable functions of* $KP\omega + (uniform\ AC)$. ***Archive for mathematical logic***, vol. 37 no. 3 (1998), pp. 199–214.

WOLFGANG BURR. *A Diller–Nahm-style functional interpretation of* $KP\omega$. Ibid., vol. 39 no. 8 (2000), pp. 599–604.

The papers under review give a characterization of the Σ-definable functions of Kripke–Platek set theory $\mathrm{KP}\omega$ in terms of set functionals of type 1 primitive recursive in $x \mapsto \omega$.

The result is established in one direction by use of a functional interpretation of KPω. The other direction is shown by use of a computability result. In the first paper, the use of a ***Dialectica***-style functional interpretation yields a restriction of the result to Σ_1-definable functions in the presence of a uniform version of the axiom of choice. In the second paper, however, Burr has refined the result by use of a *Diller–Nahm*-style functional interpretation to establish the full result.

The starting point of the papers is a result of Rathjen (*A proof-theoretic characterization of the primitive recursive set functions*, ***The journal of symbolic logic***, vol. 57 (1992), pp. 954–969), who gave a characterization of *primitive recursive set functions* by use of a subsystem of KPω. The question then arises whether it is possible to give a similar result for KPω, Kripke–Platek set theory with infinity. The answer is given in terms of set functionals of higher type.

The first paper starts with the introduction and discussion of the set functionals of higher type. In particular, a *choice functional* F_C, with $F_C a \in a$ for $\emptyset \neq a$ and $F_C \emptyset = \emptyset$ is added. Afterwards, a Hilbert-style calculus for KPω plus a uniform version of the axiom of choice (uniform AC) is given. Then, it is shown by use of a ***Dialectica***-style interpretation of KPω + (uniform AC) that the translated Σ_1 definable functions are contained in the set functionals of type 1 primitive recursive in F_C and $x \mapsto \omega$. Since KPω is based on classical logic, the interpretation is based on Shoenfield's $\forall\exists$-translation. The converse is established by a computability theorem for an extended language of the set functionals, i.e., every set functional reduces to a normal form under a suitable reduction relation. This relation contains a *conditional rewrite rule* for the union functional. Therefore, the Church–Rosser property is treated by a stratification of the reduction relation. Then the computability proof follows the standard Tait method. Since the argument can be formalized, for each separate term, in KPω + (uniform AC) the assertion is proved. In the final section, the use of the choice functional and the addition of (uniform AC) is justified by illustrating the underlying problem: The translation of (1) KP$\omega \vdash \forall x(x \neq \emptyset \rightarrow \exists y(y \in x))$ yields (2) $\forall v \exists w(v \neq \emptyset \rightarrow w \in v)$. Thus, "the 'logical, internal' choice (1) made by *KPω* becomes an external choice via the functional interpretation: any term interpreting (2) makes the choice explicit and corresponds to a kind of choice functional" (p. 213). On the other hand, to show the Σ_1-definability of such a choice functional, the uniform axiom of choice is needed for KPω.

In the second paper, Burr has substantially improved the result of the first one by replacing the ***Dialectica***-style interpretation by a *Diller–Nahm*-style interpretation. Here, the functional interpretation yields "set[s] of 'terms' containing at least one interpreting term." This requires a definition of sets of functionals of higher type which is given as follows: For any finite type τ, a pair of functionals $w^{o \rightarrow \tau}, X^o$ (o denotes a ground type) represents the "set" $\{wx \mid x \in X\}$ of functionals of type τ. Now, a modified translation of the language of KPω can be given which shows that the Σ-definable set functions of KPω coincide with the set functionals of type 1 primitive recursive in $x \mapsto \omega$ (without any further use of the choice functional).

Both papers are well written. The first one is, to a large extent, self-contained, while for the second one, of course, the first one is an indispensable prerequisite. The full result not only gives an answer to an interesting question, but it also gives an illuminating account of functional interpretations in both variants, the ***Dialectica***-style and the *Diller–Nahm*-style. In this respect, the papers seem to be very appropriate for a seminar about functional interpretation. From this point of view, the title of the first paper is chosen somewhat unfortunately: Since KPω + (uniform AC) is probably a theory without intrinsic interest, an emphasis on the functional interpretation would have given a clearer impression of the paper. On the other hand, the computability result, involving a conditional rewrite rule, is well worth noting.

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L. Beklemishev. *Another pathological well-ordering*. ***Logic Colloquium '98, Proceedings of the annual European summer meeting of the Association for Symbolic Logic, held in Prague, Czech Republic, August 9–15, 1998***, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, Lecture notes in logic, no. 13, Association for Symbolic Logic, Urbana, and A K Peters, Natick, Mass., 2000, pp. 105–108.

A construction presented in this paper is complementary to constructions (especially by G. Kreisel) showing that consistency of strong theories can be proved by a transfinite induction on a primitive recursive well-ordering of order type ω. For any constructive ordinal λ, a primitive recursive ordering E of type λ is exhibited such that the corresponding transfinite induction principle is equiconsistent with primitive recursive arithmetic PRA. Let $R(x, y)$ be a provably linear primitive recursive ordering relation of order type λ. Let

$$(R \downarrow n)(x, y) \iff R(x, y) \wedge \max(x, y) \leqq n.$$

Let $(\mathrm{TI} \downarrow n)(R)$ denote $\bigwedge_{\mathrm{GN}(A)\leqq n} \mathrm{WF}(R, A)$, where GN stands for Gödel number and

$$\mathrm{WF}(\prec, X) \iff \forall x(\forall z \prec x\ X(z) \to X(x)) \to \forall x X(x).$$

The required well-ordering E is defined by the arithmetical fixed-point lemma so that

$$\mathrm{PRA} \vdash E(x, y) \iff R(x, y) \wedge (\forall z < \max(x, y))\, \neg C(z, \mathrm{GN}(E)),$$

where $C(z, u)$ says "$u = \mathrm{GN}(E(x, y))$ and z codes a proof of a contradiction from $\mathrm{PRA}^2{+}\mathrm{TI}(E)$," and PRA^2 is a system of second-order primitive recursive arithmetic provably equiconsistent with PRA, say RCA_0.

Theorem. $\mathrm{PRA} + \mathrm{Con}(\mathrm{PRA}) \vdash \mathrm{Con}(\mathrm{PRA}^2 + \mathrm{TI}(\prec))$.

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Masahiro Hamano and Mitsuhiro Okada. *A direct independence proof of Buchholz's Hydra game on finite labeled trees*. ***Archive for mathematical logic***, vol. 37 no. 2 (1998), pp. 67–89.

The authors present a direct proof of Buchholz's $\left(\Pi^1_1\text{-CA}\right) + \mathrm{BI}$ independence result (***Annals of pure and applied logic***, vol. 33 (1987), pp. 131–155) for a labeled extension of the Kirby–Paris Hydra game for PA. The paper's aim is to connect directly the labeled Hydra game to Takeuti and Arai's finite cut-reduction approach for $\left(\Pi^1_1\text{-CA}\right) + \mathrm{BI}$ via a homomorphic embedding of the ordinal diagram order. In contrast, Buchholz's original proof dealt with infinitary cut-reductions and ordinal notations via the generalized Hardy hierarchy. Both proofs admit analogous extensions to the familiar system ID_ξ of ξ-iterated inductive definitions. Now loosely speaking, labeled *hydras* are ordinal diagrams from Takeuti's system $O(\omega + 1, 0)$, in tree form, whose roots have the minimal label 0. (Definition 5 is inaccurate, as it implies for example $\langle 0, *\#\langle 1, 0\rangle\rangle[0] = \langle 0, 0\#\langle 1, 0\rangle\rangle \notin \mathcal{H} \cup \{0\}$ instead of $\langle 0, \langle 1, 0\rangle\rangle$.) The correlated hydra term rewriting system arises by adding Buchholz's schemata of ordinal diagram rewriting rules, also called reduction rules. These reduction rules obviously decrease the ordinal diagram order, and hence the resulting reduction chains characterize certain order-descending chains of ordinal diagrams. Moreover, since $O(\omega + 1, 0)$ is well founded, all such reduction chains must terminate. The paper demonstrates that Takeuti–Arai finite cut elimination strategies in $\left(\Pi^1_1\text{-CA}\right) + \mathrm{BI}$ originally resulting in order-descending chains of ordinal diagrams, actually produce hydra reduction chains. This, in turn, implies Buchholz's main result stating that the termination property of hydra reduction chains is not provable in $\left(\Pi^1_1\text{-CA}\right) + \mathrm{BI}$.

The paper's proof is both explicit and easy, in that it shows that every ordinal diagram reduction which is determined by a Takeuti–Arai cut reduction is a composition of several hydra reductions. It is not clear, however, why arbitrary hydra reduction chains are called *strategies*. This might be sound in the underlying proof theory, where a given hypothetical derivation of (say) a numeral sequent $\rightarrow a = b$ is being gradually reduced to the cut-free derivation of $\rightarrow a = b$, by the appropriate strategy of reducing the (uniquely determined) critical cuts. Nevertheless, the correlated reduction chains of ordinal diagrams—and hence also the adjacent hydra reduction chains—do not preserve information about critical cuts. In fact, hydra strategies are so loose that it is not clear how they differ from arbitrary "natural" order-descending chains of ordinal diagrams. As a result, the mathematical advantage of hydras over the ordinary ordinal diagram approach remains unclear. It should be noted that the familiar Friedman combinatorial interpretation of proof theory of $\left(\Pi^1_1\text{-CA}\right) + \mathrm{BI}$ on the one hand is more transparent and on the other hand provides new insights into a natural homomorphic embedding order on the analogous labeled trees. The latter approach also admits extensions to generalized systems of iterated inductive definitions, by results of the reviewer (***The journal of symbolic logic***, vol. 55 (1990), pp. 157–181).

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THOMAS STRAHM. *Autonomous fixed point progressions and fixed point transfinite recursion*. ***Logic Colloquium '98, Proceedings of the annual European summer meeting of the Association for Symbolic Logic, held in Prague, Czech Republic, August 9–15, 1998***, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, Lecture notes in logic, no. 13, Association for Symbolic Logic, Urbana, and A K Peters, Natick, Mass., 2000, pp. 449–464.

In the realm of reductive proof theory, the following remarkable phenomenon has recently been observed: the classical methods of predicativity, stemming from the foundations of mathematics (Poincaré, Russell, Weyl), remain meaningfully applicable even beyond the limits established by the fundamental theorems of Schütte and Feferman in the sixties. Thus there appears a new field of investigation, the area of *meta-predicative* systems, whose full extent is still to be explored in depth.

Typically, meta-predicative systems are obtained by considering (i) theories of iterated inductive definitions where restrictions to minimal fixed points are omitted (cf. Jäger, Kahle, Setzer, and Strahm, *The proof-theoretic analysis of transfinitely iterated fixed point theories*, ***The journal of symbolic logic***, vol. 64 (1999), pp. 53–67); (ii) admissible set theories *without foundation*, which nevertheless have strong closure conditions, corresponding to recursive analogues of inaccessibility, Mahloness, etc. (cf. Jäger's pioneering *The strength of admissibility without foundation*, ***The journal of symbolic logic***, vol. 49 (1984), pp. 867–879, and Jäger and Strahm's most recent paper *Upper bounds for metapredicative Mahlo in explicit mathematics and admissible set theory*, ***The journal of symbolic logic***, vol. 66 (2001), pp. 935–958); and (iii) constructive type theories endowed with suitable notions of *universes* (in the vein of Martin-Löf, or in the sense of Feferman's explicit mathematics; cf. M. Rathjen's *The strength of Martin-Löf type theory with a superuniverse, Part I*, ***Archive for mathematical logic***, vol. 39 (2000), pp. 1–39, and Strahm's *First steps into metapredicativity in explicit mathematics*, ***Sets and proofs, Invited papers from Logic Colloquium '97—European meeting of the Association for Symbolic Logic, Leeds, July 1997***, Cambridge University Press, 1999, pp. 383–402.)

The paper under review is a valuable addition to meta-predicative proof theory and it can be seen as a natural continuation of Jäger, Kahle, Setzer, and Strahm's paper. Indeed, the central result thereof is recalled in Theorem 1, together with the natural ternary generalization $\varphi\alpha\beta\gamma$ of the standard binary Veblen φ-function and the definition of the α-times iterated fixed point theories $\widehat{\mathsf{ID}}_\alpha$ $(\omega \leqq \alpha)$, for each α below Λ_3 (= the least ordinal which cannot be named by means of φ).

A novel contribution of the paper is the definition of a powerful extension $\mathsf{Aut}(\widehat{\mathsf{ID}})$ of the transfinitely iterated fixed point theories $\widehat{\mathsf{ID}}_\alpha$. The basic idea is that we can iterate the fixed point construction not on fixed *external* primitive recursive well-orderings below Λ_3, but on orderings, which are provably well founded by proof methods *internal* to the theory $\mathsf{Aut}(\widehat{\mathsf{ID}})$ itself. This amounts to extending Peano arithmetic with two "bar rules": (i) whenever we prove that a given ordering $\lhd$ is well founded, we can argue by transfinite induction on $\lhd$ for *arbitrary formulas*; (ii) on the same hypothesis, we can assume the existence of a fixed point hierarchy along $\lhd$. The proof-theoretic ordinal of $\mathsf{Aut}(\widehat{\mathsf{ID}})$ is $\varphi 200$ (in the extended Veblen hierarchy), and hence much larger than the ordinals involved in Jäger, Kahle, Setzer, and Strahm's paper.

The paper also contributes to the study of subsystems of analysis, based on *fixed point transfinite recursion* FTR, i.e., the principle that, for each (arithmetical) inductive operator form $\mathcal{A}(X, Y, x, a)$ and arbitrary well-ordering X, there exists a *transfinite hierarchy of fixed points*, which are generated by $\mathcal{A}$ along X. Of course, FTR can be regarded as a kind of axiomatic internal version of the rules of $\mathsf{Aut}(\widehat{\mathsf{ID}})$. The strength of fixed point recursion is characterized by the following results: (1) The system FTR_0 with the axiom of number-theoretic induction and $\mathsf{Aut}(\widehat{\mathsf{ID}})$ have the same proof-theoretic strength. (2) Its variant FTR with full number-theoretic schema is slightly stronger, having ordinal $\varphi 20\varepsilon_0$.

The computation of the upper bounds interestingly involves a detour through a set-theoretic meta-predicative system KPh^0, which comprises the axioms of admissible sets *without foundation*, but also assumes that the universe of sets is *hyperinaccessible* (roughly, a universe I of sets is inaccessible provided it is an "admissible limit" of admissible sets, while a set universe U is "hyperinaccessible" if it is an "admissible limit" of inaccessibles).

According to work of Jäger and Strahm (cited above) on the proof theory of recursively Mahlo universes, the proof-theoretic ordinal of KPh^0 is again $\varphi 200$. Therefore the proof-theoretic ordinal of FTR_0 is derived by defining a natural embedding into KPh^0 (cf. Theorem 7). As to the system with full number-theoretic induction FTR, it is interpreted into a suitable variant of KPh^0 of strength $\varphi 20\varepsilon_0$.

The concluding section offers a survey of related subsystems of analysis and set theory, and includes a brief discussion of the connection between meta-predicativity, Feferman's explicit mathematics, and Martin-Löf's type theory. In this connection, there emerges the important role of the notion of *universe* and of its strengthenings.

To sum up, the author is able to offer a lucid and informative picture of recent developments in meta-predicative proof theory.

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B. È. Shapirovskiĭ. *O π-kharaktere i π-vese v bikompaktakh*. ***Doklady Akademii Nauk SSSR***, vol. 223 (1975), pp. 799–802.

B. È. Šapirovskiĭ. *On π-character and π-weight in compact Hausdorff spaces*. English translation by I. Juhász of the preceding. ***Soviet mathematics***, vol. 16 no. 4 (for 1975, pub. 1976), pp. 999–1004.

B. È. Shapirovskiĭ. *Ob otobrazheniyakh na Tikhonovskie kuby*. ***Uspekhi matematicheskikh nauk***, vol. 35 no. 3 (1980), pp. 122–130.

B. E. Shapirovskii. *Maps onto Tikhonov cubes*. English translation by A. West of the preceding. ***Russian mathematical surveys***, vol. 35 no. 3 (1980), pp. 145–156.

In his important 1975 paper, the author proves that a countably tight ($x \in \overline{A}$ implies $x \in \overline{N}$ for some countable $N \subset A$) compact Hausdorff space has countable π-character (i.e., for every point x there is a countable sequence $\langle B_n \rangle_{n\in\omega}$ of non-empty open sets—not necessarily containing x—such that every neighborhood of x contains a B_n). It is also proved that a hereditarily normal compact space has a dense set of points with countable π-character. Proposition 2 in the proof of the latter result deserves particular attention.

The method of proof was later extended by the author in his 1980 paper to a complete characterization, with the help of the π-character, of the existence of a continuous map of a compact Hausdorff space onto a Tychonoff cube of I^κ for every given infinite cardinal κ.

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A. V. ARKHANGEL′SKIĬ. *O moshchnosti bikompaktov c pervoĭ aksiomoĭ schetnosti*. ***Doklady Akademii Nauk SSSR***, vol. 187 (1969), pp. 967–970.

A. V. ARHANGEL′SKIĬ. *On the cardinality of bicompacta satisfying the first axiom of countability*. English translation by Z. Skalsky of the preceding. ***Soviet mathematics***, vol. 10 (1969), pp. 951–955.

R. POL. *Short proofs of two theorems on cardinality of topological spaces*. English with Russian summary. ***Bulletin de l'Académie Polonaise des Sciences***, Série des sciences mathématiques, astronomique et physiques, vol. 22 (1974), pp. 1245–1249.

ALAN DOW. *An introduction to applications of elementary submodels to topology*. ***Topology proceedings*** (Auburn), vol. 13 (1988), pp. 17–72.

In their famous work on bicompact spaces (***Bulletin international de l'Académie Polonaise des Sciences et des Lettres***, Classe des sciences mathématiques et naturelles, Série A, Sciences mathématiques, année 1923 (pub. 1924), pp. 5–8), P. S. Aleksandrov and P. S. Urysohn posed the problem whether all first countable compact Hausdorff spaces have no more than 2^ω points. In the paper by A. V. Arhangelski the author proves that the answer is yes, in fact, compact (every open cover has a finite subcover) can be relaxed to Lindelöf (every open cover has a countable subcover) and the result generalizes to higher cardinals. This celebrated result has become one of the milestones in set-theoretic topology.

Its original, complicated proof underwent extensive simplification over the years. Following several attempts by various authors, R. Pol gives a simple saturation-type proof.

The contemporary proof by A. Dow goes by elementary submodels and is pleasantly short.

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RANDALL DOUGHERTY and MATTHEW FOREMAN. *Banach–Tarski paradox using pieces with the property of Baire*. ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 89 (1992), pp. 10726–10728.

RANDALL DOUGHERTY and MATTHEW FOREMAN. *Banach–Tarski decompositions using sets with the property of Baire*. ***Journal of the American Mathematical Society***, vol. 7 (1994), pp. 75–124.

One of the more intriguing areas of research into the foundations of the real numbers concerns the relative strength of the assertions LM ("all sets of reals are Lebesgue measurable") and PB ("all sets of reals have the property of Baire"). Recall that the measurable sets are the smallest σ-algebra containing the Borel sets and the sets of measure zero, while the property of Baire sets are the smallest σ-algebra containing the Borel sets and the nowhere dense sets. The results I am referring to are $\mathrm{Con(ZF)} \Leftrightarrow \mathrm{Con(ZF + PB)}$ and Con(ZF + inaccessible cardinal) $\Leftrightarrow \mathrm{Con(ZF + LM)}$. These important theorems show that the Baire and measure families are very different: one has an intimate relationship with large cardinals while the other does not.

The amazing and unexpected results of the papers under review add to our appreciation of the differences between the two families. The classic Banach–Tarski paradox, a construction that is surely destined for immortality, shows that, assuming the axiom of choice, one can take apart a sphere S in 3-space into pieces A, B, C, and D so that A and B can be rotated to form S, and the same is true of C and D. (The paradox holds for a solid ball too, though with five pieces, not four). Of course, the pieces cannot be measurable, because $1 + 1 \neq 1$.

In 1930, E. Marczewski asked the intriguing question: Can the pieces in the Banach–Tarski paradox have the property of Baire? Dougherty and Foreman give an affirmative answer to the question, using six pieces for the sphere (or for the ball). As noted, this emphasizes the difference between the measurable sets and the Baire sets. Their proof, though intricate, is not too difficult, and all the details can be found in the 1992 paper; the longer 1994 article contains a comprehensive discussion of the connection of their construction to diverse areas of set theory and measure theory.

One striking result they obtain is a version that does not require the axiom of choice: If A and B are non-empty bounded open subsets of $\mathbb{R}^n$ ($n \geqq 3$), then there are finitely many open subsets of A that can be rearranged by using rigid motions to form a dense open subset of B.

Questions arising from the Banach–Tarski paradox have led to several fascinating results in the 1990's, results that are no less surprising than the original paradox. Indeed, at the end of the 1994 paper, the authors observe that most of their main results are "highly implausible." The construction of Dougherty and Foreman, together with Laczkovich's surprising positive solution of Tarski's circle-squaring problem (see *Paradoxical decompositions: a survey of recent results*, ***First European Congress of Mathematics, Paris, July 6–10, 1992*** Volume II, Birkhäuser, 1994, pp. 159–184), are the two most important recent results related to paradoxical (in the case of the former) and non-paradoxical (for the latter) decompositions of sets.

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JAMES E. BAUMGARTNER. *On the size of closed unbounded sets*. ***Annals of pure and applied logic***, vol. 54 (1991), pp. 195–227.

Suppose that κ is a regular, uncountable cardinal and let $\mathfrak{A}$ be a structure of cardinality $\lambda > \kappa$ for a countable language. How many substructures of cardinality $< \kappa$ can $\mathfrak{A}$ have? This question can be reformulated in terms of a generalized notion of closed, unbounded (CUB) set: Let $[\lambda]^{<\kappa}$ be the set of subsets of λ with cardinality $< \kappa$. If $\mathfrak{A}$ is a structure for a countable language with universe λ, we let $C(\mathfrak{A})$ denote the set of universes of substructures of $\mathfrak{A}$. A *CUB subset* of $[\lambda]^{<\kappa}$ is a set of the form $C(\mathfrak{A}) \cap [\lambda]^{<\kappa}$ for some $\mathfrak{A}$. Thus our question becomes: How large are CUB subsets of $[\lambda]^{<\kappa}$?

An obvious lower bound is λ and an obvious upper bound is $\lambda^{<\kappa}$, the cardinality of the entire set $[\lambda]^{<\kappa}$. Of course if the cofinality of λ is at least κ and the generalized continuum hypothesis (GCH) holds, then these cardinals are the same; but in the present context we do not wish to assume the GCH.

In the paper under review, Baumgartner treats the case $\lambda = \kappa^{+n}$ for some finite $n > 0$. He first considers the stationary sets

$$S(\kappa, \lambda; \kappa_0, \dots, \kappa_n) = \{x \in [\lambda]^{<\kappa} \mid \sup(x \cap \kappa^{+i}) \text{ has cofinality } \kappa_i \text{ for all } i \leqq n\},$$

where $\kappa_0, \dots, \kappa_n < \kappa$ are regular cardinals. These sets are disjoint and have union equal to a CUB set of maximal cardinality $\lambda^{<\kappa}$. He shows that if none of the κ_i's is ω, then the intersection of some CUB set with $S(\kappa, \lambda; \kappa_0, \dots, \kappa_n)$ has (the minimum possible) cardinality λ. The idea behind this proof is as follows: He shows that for an appropriate CUB set C, elements of $S = C \cap S(\kappa, \lambda; \kappa_0, \dots, \kappa_n)$ must be ω-closed, in the sense that they contain all of their limit points of cofinality ω. It follows that each element x of S is uniquely determined by its supremum together with its intersection with κ^{+n-1}, as any two ω-closed subsets of the supremum of x have an unbounded intersection, and we may assume that for α in x, x is closed under a bijection between α and the cardinality of α. Now by induction on n it follows that for x in the intersection of some CUB set with $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$, x is uniquely determined by the suprema of its intersections with the κ^{+i}, for which there are only λ possibilities.

It follows that any lower bound on the size of CUB sets must turn on their intersection with $S(\kappa, \lambda; \kappa_0, \ldots, \kappa_n)$ when one of the κ_i's is ω. Baumgartner in fact obtains the lower bound of λ^ω, in the case where κ_i is ω for some $i > 0$, and therefore can conclude that CUB subsets of $[\lambda]^{<\kappa}$ always have cardinality at least λ^ω. The proof of this latter fact is based on a tree construction: For a given function $f : [\lambda]^{<\omega} \to \lambda$, Baumgartner builds a tree on λ of height ω in which every node has λ immediate successors, and such that if σ, τ are distinct immediate successors to a common node of the tree, then τ does not belong to the closure under f of any infinite branch through σ. Thus there are λ^ω distinct subsets of λ, each closed under f. As any CUB subset of $[\lambda]^{<\kappa}$ consists of the f-closed subsets of λ for some f, this gives the desired result.

Can this be improved to λ^{ω_1} when κ is greater than ω_1 (and λ^{ω_1} is greater than λ^ω)? Baumgartner shows that by starting with an ω_1-Erdős cardinal in a model satisfying a version of Jensen's $\Box$ principle, one can force to obtain a model in which $\omega_3^\omega < \omega_3^{\omega_1}$ and every CUB subset of $[\omega_3]^{<\omega_2}$ has cardinality $\omega_3^{\omega_1}$. The forcing is simply the Levy collapse of the ω_1-Erdős cardinal to ω_3, followed by forcings to add Cohen subsets of ω_1. Baumgartner builds a tree on λ of height ω_1 in the generic extension, now using indiscernibility to separate the closure of distinct branches. Thus the answer to our question is "Yes" if we assume the consistency of an ω_1-Erdős cardinal. (Baumgartner shows how to force $\Box$ while preserving the ω_1-Erdős property; thus only the consistency of an ω_1-Erdős cardinal is needed.) Results of Magidor (JSL LX 701) show that this use of ω_1-Erdős cardinals is in fact necessary.

There are many related results in Baumgartner's paper, which is beautifully written, and highly recommended for those with an interest in combinatorial set theory.

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James E. Baumgartner. *Generic graph construction*. ***The journal of symbolic logic***, vol. 49 (1984), pp. 234–240.

Matthew Foreman and Richard Laver. *Some downwards transfer properties for* $\aleph_2$. ***Advances in mathematics***, vol. 67 (1988), pp. 230–238.

Saharon Shelah. *Incompactness for chromatic numbers of graphs*. ***A tribute to Paul Erdős***, edited by A. Baker, B. Bollobás, and A. Hajnal, Cambridge University Press, Cambridge, New York, and Oakleigh, Victoria, 1990, pp. 361–371.

The problem addressed by these four preëminent mathematicians originates from Erdős and Hajnal. They asked if there exists, under GCH, a graph with size and chromatic number $\aleph_2$, all of whose smaller subgraphs are countably chromatic (*On chromatic number of infinite graphs*, ***Theory of graphs, Proceedings of the colloquium held at Tihany, Hungary, September 1966***, Akadémiai Kiadó and Academic Press, 1968, pp. 83–98). The reason for this particular choice of parameters is that they had proved that (under CH) there exists a graph of cardinality $\aleph_2$, chromatic number $\aleph_1$, with every smaller subgraph countably chromatic. The question was repeated (as Problem 41(B)) in the famous problem list of Erdős and Hajnal (*Unsolved problems in set theory*, ***Axiomatic set theory***, Proceedings of symposia in pure mathematics, vol. 13 part 1, American Mathematical Society, 1971, pp. 17–48). From there it obtained considerable fame, especially since, although Shelah quickly solved the analogous case for the coloring number, not much happened for more than a decade.

It was then solved in the eighties and, although the pattern follows the classical scheme (consistency of an example, consistency of no example, example in L), the methods invented are several levels deeper than usual. All three proofs have some remarkable elements. Problems on the chromatic number are simply hard, usually.

In his characteristically well-written paper, Baumgartner proves the consistency of an example, i.e., that over a model of CH one can force (with a countably closed $\aleph_2$-c.c. forcing) for a graph on ω_2 with chromatic number $\aleph_2$ that every smaller subgraph is countably chromatic. He starts with the naïve forcing in which a condition gives the graph on a countable subset a of ω_2 along with the corresponding values of the good colorings of the initial segments $f_\alpha : \alpha \to \omega$, that is, f_α is determined on $a \cap \alpha$ for $\alpha \in a$. This approximation will not work, as the usual argument for the $\aleph_2$-c.c. cannot be carried out. The author's remedy is to assign to each $\alpha, \beta \in a$ an infinite, coinfinite set of colors $A(\alpha, \beta) \subseteq \omega$ and then to require that if $\gamma < \alpha, \beta$ and γ is joined to α then $f_\beta(\gamma) \in A(\alpha, \beta)$. This saves the chain condition, but the hope for one-edge amalgamation (vital for showing that the chromatic number is $\aleph_2$) is lost. To mend the situation, Baumgartner reorganizes the construction considerably. First he concentrates the whole graph on the set $W = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$. Then he breaks $A(\alpha, \beta)$ into the infinite pieces $B(\alpha, \beta, 0), B(\alpha, \beta, 1), B(\alpha, \beta, 2), \ldots$ and requires that if $\gamma < \alpha$ is joined to α and $f_\beta(\gamma) \in B(\alpha, \beta, n)$ then γ be higher than the nth term of a predetermined sequence converging to α. That is, colors are forbidden to be used up "too early," and so the one-edge-amalgamation situation can be arranged so that there is some unused color at the point it is needed. Still, considerable skill is needed to execute these ideas, as the above requirements radically change the Δ-system lemmas we can use.

In the second paper under review, Foreman and Laver prove that if it is consistent that there exists a huge cardinal then it is also consistent that every graph of size and chromatic number $\aleph_2$ contains a subgraph of size and chromatic number $\aleph_1$. In fact they produce a model in which $\aleph_2$ has strong properties of compactness. Their method is a powerful variant of the cardinal collapse techniques introduced by K. Kunen. If κ is huge, $j : V \to M$ is an elementary embedding witnessing this, $j(\kappa) = \lambda$, then a certain iteration $P * Q$ is applied where P is a preliminary iteration of length κ making $\kappa = \omega_1$, and then Q further collapses λ to ω_2. In proofs like this we usually have that $j(Q)$ is very closed and $P * Q$ nicely embeds into $j(P)$. What we have here is that if X is a λ-chromatic graph on λ (in V^{P*Q}) then $j[X]$ will stay λ-chromatic even in $M^{j(P)*j(Q)}$ as $j(Q)$ and M are sufficiently closed and $j(P)/(P * Q)$ is κ-centered. This latter property (which is so strong that I simply did not believe it at first reading) is obtained by a certain nested iteration of the Silver collapse, the key element of the proof of the centeredness property roughly being the well-known lemma that the topological product of $2^{\aleph_0}$ separable spaces is separable. The authors point out several applications of their model; I foresee more, especially in partition calculus. An intriguing feature is that hugeness seems to be essential in the proof; apparently Magidor's trick does not work for scaling it down to almost hugeness.

In the third paper, Shelah proves that in L there exist examples, that is, there is a graph with size and chromatic number $\aleph_2$ such that every smaller subgraph is countably chromatic. He uses the so-called "diamond-in-the-square principle," a combination of the diamond and the box principles, discovered and applied by U. Abraham, S. Shelah, and R. M. Solovay (*Squares with diamonds and Souslin trees with special squares*, ***Fundamenta mathematicae***, vol. 127 (1986), pp. 133–162). Although Shelah's argument is rather short and straightforward (and no doubt, he discovered it in less time than one can say "pcf theory"), it is tricky, it does not work when one omits any of the apparently superfluous elements. It is quite strange that although Baumgartner uses a cleverly modified Prikry-type forcing, Shelah's argument is well below the morass level. Shelah immediately generalizes his theorem to give that in L

if $\kappa > \omega_1$ is a regular, not weakly compact cardinal, then there is a κ-chromatic graph on κ, all of whose smaller subgraphs are countably chromatic. There are several other results in Shelah's paper, which are not described here.

In closing, I mention that the following problem seems to be open. Is it consistent that every graph of size $\aleph_2$ and uncountable chromatic number contains a subgraph of size $\aleph_1$ and uncountable chromatic number? Of course, CH must fail, by the Erdős–Hajnal example we started with.

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GREG HJORTH and ALEXANDER S. KECHRIS. *Analytic equivalence relations and Ulm-type classifications*. ***The journal of symbolic logic***, vol. 60 (1995), pp. 1273–1300.

GREG HJORTH, ALEXANDER S. KECHRIS, and ALAIN LOUVEAU. *Borel equivalence relations induced by actions of the symmetric group*. ***Annals of pure and applied logic***, vol. 92 (1998), pp. 63–112.

Let E be an analytic equivalence relation on a Polish space X or on a standard Borel space X, that is, a space equipped with the σ-algebra of all Borel sets with respect to some Polish topology on X. One way of assessing complexity of E is to assign to its equivalence classes as simple invariants as possible using definable functions. (By assignment of complete invariants one understands assigning objects to elements of X so that two elements are mapped to the same object precisely when they are E-equivalent.) These invariants may be natural numbers or reals (members of 2^ω) but may be more complicated. In most situations, the invariants turn out to be themselves equivalence classes of some other, better understood, equivalence relation defined on a Polish or standard Borel space. For example, reals can be viewed as equivalence classes of equality on 2^ω; more complicated invariants arise, naturally, from more complex equivalence relations. This approach, therefore, boils down to comparing E with other known equivalence relations by using the following notion of reducibility. E is *reducible to* an equivalence relation F defined on a Polish, or standard Borel, space Y if there exists a function $f : X \to Y$ such that $x_1 E x_2$ if and only if $f(x_1) F f(x_2)$. (Members of Y/F are then invariants for E. The function f induces an injection from X/E to Y/F.) Obviously, the above statement is interesting only if f is assumed to be in some sense simple, say Borel or $\mathbf{\Delta}^1_2$, etc. The level of definability suitable here has to be adjusted according to the problem at hand. In symbols one writes $E \leqq_\Gamma F$ where Γ indicates the level of definability of f; for example, $E \leqq_{\mathrm{B}} F$ or $E \leqq_{\mathbf{\Delta}^1_2} F$ if f is Borel or $\mathbf{\Delta}^1_2$, respectively.

We denote the equivalence relation of equality on a set A by $\Delta(A)$. Equivalence relations that admit reals as complete invariants, that is, equivalence relations that are $\leqq_{\mathrm{B}} \Delta(2^\omega)$, are called smooth.

Two fundamental dichotomy theorems hold for Borel equivalence relations. The first one is due to J. Silver (*Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, JSL LII 869) and shows that for a Borel, in fact even $\mathbf{\Pi}^1_1$, equivalence relation E either $\Delta(2^\omega) \leqq_{\mathrm{B}} E$ or $E \leqq_{\mathrm{B}} \Delta(\omega)$. The second dichotomy, due to L. Harrington, A. S. Kechris, and A. Louveau (*A Glimm–Effros dichotomy for Borel equivalence relations*, JSL LXIII 749), says that if E is a Borel equivalence relation, then either E is smooth or else a canonical non-smooth equivalence relation Borel reduces to E. In symbols, $E \leqq_{\mathrm{B}} \Delta(2^\omega)$ or $E_0 \leqq_{\mathrm{B}} E$ where E_0 is defined on 2^ω and makes two binary sequences equivalent if they are eventually equal. The combination of these two dichotomy theorems provides a classification, up to mutual Borel reducibility, of the bottom part of $\leqq_{\mathrm{B}}$ among Borel equivalence relations: a Borel equivalence relation is above E_0 or is mutually Borel reducible to $\Delta(2^\omega)$, $\Delta(\omega)$, or $\Delta(n)$ for $n \in \omega$.

Since equivalence relations encountered in applications are frequently analytic, and not always Borel, it is natural to inquire if the above results generalize to analytic equivalence relations. It was quickly realized that there exist analytic equivalence relations with precisely ω_1 equivalence classes, so Silver's theorem fails in this generality. J. P. Burgess, however (*Effective enumeration of classes in a Σ^1_1 equivalence relation*, ***Indiana University mathematics journal***, vol. 28 (1979), pp. 353–364), found an appropriate modification of Silver's dichotomy and, under the assumption that each real has a sharp, established the modified dichotomy for analytic equivalence relations.

In *Analytic equivalence relations and Ulm-type classifications*, Hjorth and Kechris take up the problem of carrying over the more subtle Harrington–Kechris–Louveau dichotomy, in an appropriate form, to analytic equivalence relations. Corollary 6.3 in the paper exhibits a natural example for which the dichotomy, as stated for Borel equivalence relations, fails in the more general context of analytic equivalence relations. For a fixed prime number p, let P be the subset of 2^{ω^3} consisting of all $x \subseteq \omega^3$ such that x as a ternary relation on ω is the graph of a multiplication with which ω becomes a p-group. Two such x and x' are equivalent if a certain permutation of ω transfers x to x' when applied coordinatewise. This means simply that the two groups with multiplication given by x and x' are isomorphic. (It is a particular instance of a logic action of S_∞—the Polish group of all permutations of ω.) The authors prove, assuming that ω_1 is inaccessible in the constructible universe, that neither $E \leqq_{\mathbf{\Delta}^1_2} \Delta(2^\omega)$ nor $E_0 \leqq_{\mathbf{\Delta}^1_2} E$ where E is the isomorphism equivalence relation on P defined above. Fortunately, the same example suggests what type of invariants, necessarily more complicated than elements of 2^ω, one may hope to find in the case when E_0 does not reduce to an equivalence relation. Ulm's theorem (L. Fuchs, ***Infinite abelian groups***, Volume II, Academic Press, 1973, Theorem 77.3) can be viewed abstractly as assigning to each element x of P a member $U(x)$ of $2^{<\omega_1}$ (the bounded subsets of ω_1) so that x and x' are isomorphic precisely when $U(x) = U(x')$, so the assignment U provides complete invariants for the isomorphism equivalence relation. Moreover, it is known (see J. Barwise and P. Eklof, *Infinitary properties of abelian torsion groups*, ***Annals of mathematical logic***, vol. 2 no. 1 (1970), pp. 25–68) that one can find such an assignment which is additionally definable: $\mathbf{\Delta}^1_2$ or even C-measurable in the following sense. (Recall that a function is C-measurable if the pre-image of each Borel set belongs to the smallest σ-algebra containing all Borel sets and closed under Suslin's operation $\mathcal{A}$.) Let $WO \subseteq \omega^\omega$ be the set of all codes of countable well-orderings. One produces a mapping $U^* : P \to 2^\omega \times 2^\omega$ in such a way that $U^*(x) \in WO \times 2^\omega$ and $U(x) = [U^*(x)]$ fulfills the above stipulation imposed on the function U. Here for $(x, a) \in WO \times 2^\omega$, $[(x, a)]$ is the subset of the ordinal isomorphic to the well-ordering coded by x that is the image of a via the isomorphism. This U^* is now $\mathbf{\Delta}^1_2$ or C-measurable.

The paper is concerned with proving the new type of dichotomy for various classes of analytic equivalence relations. Let me restate the two cases of the dichotomy:

(I) $E_0 \leqq_{\mathrm{B}} E$. (II) There exists $U : X \to 2^{<\omega_1}$ such that xEx' if and only if $U(x) = U(x')$.

In full generality when E is assumed to be only analytic, it is proved, with the assumption that each real has a sharp, that the dichotomy holds and U can be assumed to be $\mathbf{\Delta}^1_2$ in the sense explained above (Theorem 5.4). The authors also obtain very interesting strengthenings of this result if more is assumed about E. If, in addition to E's being analytic, each equivalence class of E is Borel, then the assumption of the existence of sharps can be dropped (Theorem 4.4).

Particularly important types of equivalence relations are the ones induced by Borel actions of Polish groups on standard Borel or Polish spaces. If a Polish group G acts on X, we define the induced *orbit equivalence relation* by letting $x \in X$ and $y \in X$ be equivalent if $x = gy$ for some $g \in G$. Each such equivalence relation is analytic and has Borel equivalence classes. Now if E is assumed to be the orbit equivalence relation of a Borel action of a Polish group, then in the dichotomy U can be assumed to be C-measurable rather than merely $\mathbf{\Delta}^1_2$ (Theorem 2.1). This case encompasses the situation described in the p-group example: the isomorphism equivalence relation on P is the orbit equivalence relation of a continuous action of the Polish group S_∞ described above. Furthermore, if the Polish group is assumed to be Abelian, the authors get an even stronger result: in (II) the set $2^{<\omega_1}$ can be replaced by 2^ω and U is C-measurable. (This was subsequently improved in G. Hjorth and S. Solecki, *Vaught's conjecture and the Glimm–Effros property for Polish transformation groups*, ***Transactions of the American Mathematical Society***, vol. 351 (1999), pp. 2623–2641, and H. Becker, *Polish group actions: dichotomies and generalized elementary embeddings*, ***Journal of the American Mathematical Society***, vol. 11 (1998), pp. 397–449.)

Borel actions of the Polish group S_∞ are particularly interesting to logicians since among them are the so-called *logic actions*. One is given a countable language L and considers models with universe ω for this language. The set of all such models is equipped with a natural topology and forms a Polish space X_L. Then S_∞ action on X_L is done by permuting domains of models in X_L. Two models are in the same orbit of the action precisely when they are isomorphic. This is a continuous action of S_∞ on the Polish space X_L with the orbit equivalence relation coinciding with the equivalence relation of isomorphism between models. (The details are handled in a manner described above for p-groups.) A classical theorem of López-Escobar says that each Borel subset of X_L invariant under the action of S_∞ is the set of all models of a sentence from $L_{\omega_1\omega}$ and vice versa. Therefore, the study of isomorphism between countable models of a sentence from $L_{\omega_1\omega}$ is essentially equivalent to studying the restrictions of the logic action of S_∞ to invariant Borel subsets of X_L. In this special situation, the authors recast (II) of the above dichotomy in fully model-theoretic terms. Let us fix a language L. Let σ_ξ be an $L_{\omega_1\omega}$ sentence fulfilled by precisely those countable models whose Scott height is $< \xi$. Hjorth and Kechris show that if σ is an $L_{\omega_1\omega}$ sentence, then the equivalence relation of isomorphism on the countable models of σ has invariants of the type described in (II) precisely when for each $\xi < \omega_1$, the isomorphism equivalence relation on countable models of $\sigma \wedge \sigma_\xi$ is smooth, which is further equivalent to the following purely model-theoretic statement: let T be a complete theory containing σ and σ_ξ; if each countable subset of T has a model, then T has a countable model. This is also equivalent to the same statement with "T has a model" replaced by "T has a countable model."

In the second paper under review, *Borel equivalence relations induced by actions of the symmetric group*, the authors study orbit equivalence relations induced by Borel actions of closed subgroups of S_∞ in the case where these equivalence relations are Borel. The assumption of Borelness of the orbit equivalence relation has a natural interpretation in the case of the restriction of the logic action of S_∞ on X_L, for some countable language L, to an invariant Borel set, that is, to the set of all models of a sentence from $L_{\omega_1\omega}$. As proved by H. Becker and A. S. Kechris (***The descriptive set theory of Polish group actions***, JSL LXIII 1604, Corollary 7.1.4), if B is a Borel invariant subset of X_L, then the orbit equivalence relation on B is Borel precisely when the Scott heights of all models in B are bounded below ω_1.

The paper investigates descriptive set-theoretic complexity of such equivalence relations and possible complete invariants that can be used in classifying them. As the authors point out, there are obvious problems with assessing descriptive set-theoretic complexity of an equivalence relation E. For example, the standard Borel space on which E is defined need not come equipped with a Polish topology and different Polish topologies may generate the same Borel structure but give different complexity to E. Therefore, the following notion serves best as a formal definition that captures the descriptive complexity of E. Let Γ be a Wadge class. An equivalence relation E on a standard Borel space is called *potentially* Γ

if there exists a Polish topology τ on X whose family of Borel sets coincides with the Borel structure on X and such that E is in Γ if $X \times X$ is considered with the product topology $\tau \times \tau$. We say that the *potential class of* E *is* Γ if E is potentially Γ and it is not potentially Γ' for any Wadge class Γ' properly included in Γ. There is a simple relationship between this notion and Borel reducibility between equivalence relations: if the potential class of F is Γ and $E \leqq_{\mathrm{B}} F$, then the potential class of E is included in Γ.

The main result of the paper pinpoints the Wadge classes that are potential classes of Borel orbit equivalence relations induced by Borel actions of closed subgroups of S_∞ on standard Borel spaces (Theorem 1 or Theorems 3.1, 4.1, 4.2). They are $\mathbf{\Pi}^0_\lambda$, $\mathbf{\Sigma}^0_{\lambda+1}$, $\mathbf{\Pi}^0_{\lambda+n+1}$, the differences of two $\mathbf{\Pi}^0_{\lambda+n+1}$ sets where $n \geqq 1$ and $\lambda = 1$ or λ is limit, $\mathbf{\Delta}^0_1$, or $\bigoplus_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$, for λ limit. (The class $\bigoplus_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$ consists of sets of the form $\bigcup_n A_n$ for which $A_n \in \bigcup_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$ and there exists a clopen partition $\bigcup_n X_n = X$ of X such that $A_n \subseteq X_n$ for each n.)

Each of these classes is realized as a potential class of the Borel orbit equivalence relation induced by a continuous action of S_∞ on a Polish space (Theorem 5.8).

The authors uncover a connection between being potentially Γ and the Borel reducibility order among Borel equivalence relations (Theorems 2 and 3 or Corollary 6.4). Namely, for each of the above classes Γ there exists a Borel equivalence relation E_Γ that is induced by a continuous action of S_∞ on a Polish space and is such that if E is a Borel orbit equivalence relation induced by a Borel action of a closed subgroup of S_∞ on a standard Borel space, then E is potentially Γ if and only if $E \leqq_{\mathrm{B}} E_\Gamma$.

The equivalence relations E_Γ for $\Gamma = \mathbf{\Pi}^0_\lambda$ and $\mathbf{\Pi}^0_{\lambda+n+1}$ for $\lambda = 1$ or limit and $n \geqq 1$, denoted by $\cong_\lambda$ and $\cong_{\lambda+n}$, respectively, turn out to be the very natural equivalence relations considered already in a different situation by H. Friedman and L. Stanley (*A Borel reducibility theory for classes of countable structures*, ***The journal of symbolic logic***, vol. 54 (1989), pp. 894–914). $\cong_{\lambda+n}$ is essentially equality on ω if $\lambda + n = 0$ and is equality on the set obtained from ω by applying the operation of taking countable subsets $\lambda + n$ times. The equivalence relation E_Γ for $\Gamma = \bigoplus_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$ is obtained in the obvious fashion from $\cong_\alpha$ for $\alpha < \lambda$. E_Γ for $\Gamma =$ the differences of two $\mathbf{\Pi}^0_{\lambda+n+1}$ sets or $\Gamma = \mathbf{\Sigma}^0_{\lambda+1}$, where $n \geqq 1$ and $\lambda = 1$ or λ is limit, are subtle modifications of $\cong_\alpha$'s. Via codings, all these equivalence relations can be viewed as orbit equivalence relations of the logic action of S_∞ on the family of all models with universe ω of a sentence of $L_{\omega_1\omega}$ for some language L depending on Γ.

The proof that the Wadge classes mentioned in the theorem are realized as the potential classes of orbit equivalence relations of continuous actions of S_∞ on Polish spaces involves a construction with other consequences of independent interest. Call a subgroup H of a Polish group G *Polishable* if it carries a Polish group topology whose Borel sets coincide with Borel subsets of H inherited from G. For each Γ mentioned above, except necessarily $\bigoplus_{\alpha<\lambda} \mathbf{\Pi}^0_\alpha$, the authors construct a Polishable subgroup H of 2^ω such that H is precisely of class Γ and, with its unique Polish group topology, can be embedded as a closed subgroup of S_∞ (Lemmas 5.4–5.6). Apart from the main application in the proof that appropriate Wadge classes are realized as potential classes of the orbit equivalence relations of continuous actions of S_∞, the authors derive from the existence of these Polishable subgroups three additional consequences: the results of Friedman–Stanley ($\cong_\beta \nleqq_{\mathrm{B}} \cong_\alpha$ for $\alpha < \beta < \omega_1$, op. cit.), Harrington (there is no Borel equivalence relation E with $\cong_\alpha \leqq_{\mathrm{B}} E$ for all $\alpha < \omega_1$), and Solecki (there is a Polish Abelian group with a continuous action on a Polish space with non-Borel orbit equivalence relation, *Equivalence relations induced by actions of Polish groups*, ***Transactions of the American Mathematical Society***, vol. 347 (1995), pp. 4765–4777).

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Matthew Foreman. *A descriptive view of ergodic theory*. ***Descriptive set theory and dynamical systems***, edited by M. Foreman, A. S. Kechris, A. Louveau, and B. Weiss, London Mathematical Society lecture note series, no. 277, Cambridge University Press, Cambridge, New York, etc., 2000, pp. 87–171.

Alexander S. Kechris. *Descriptive dynamics*. Ibid., pp. 231–258.

I will begin with the shorter of the two papers, which is the one by Kechris, and is organized into four expository lectures.

The title for this paper is already interesting. It is a term Kechris has formulated to refer to some of the recent work in descriptive set theory on Polish group actions. The "dynamics" comes from the continuous, Borel, or simply definable, action of a Polish group; the "descriptive" from the application of the techniques of descriptive set theory. This area, under the guise of "invariant descriptive set theory," finds its roots in papers by Vaught, and underwent a renaissance following Becker and Kechris's ***The descriptive set theory of Polish group actions*** (JSL LXIII 1604).

Perhaps the simplest way to understand this subject is in terms of theorems that were proved classically and only with much greater difficulty generalized to the invariant context. Thus we have in mind to restrict ourselves to Polish spaces on which some Polish group acts in either a Borel or continuous fashion.

As for notation, a *Polish group* is a topological group that is Polish as a space—that is to say, separable and admitting a complete metric. We say that X is a *Polish G-space* if the Polish group G acts continuously on the Polish space X.

For instance we know classically that every Polish space can be realized as a G_δ subset of the Hilbert cube, $([0,1])^{\mathbb{N}}$. The analogue of this theorem in the invariant context should be a result to the effect that for any Polish group we can find a *universal* Polish G-space into which all other Polish G-spaces may be somehow reduced. Lecture I of Kechris's paper recalls various theorems and counterexamples, depending on whether we insist that the G-embedding be continuous or merely Borel, the space compact or merely Polish.

Classically, Kuratowski showed that in any Polish space X and for any Borel $B \subset X$ we may find a richer Polish topology on X in which B now becomes clopen. Although the situation is, as might be anticipated, more subtle in the case of a continuous action of a Polish group, Becker and Kechris proved various extensions to the context of Polish group actions: For instance, if G is a Polish group and X a Polish G-space, then any *invariant* $(g \cdot B = B$ for all $g \in G)$ Borel set B may be converted into a clopen one, maintaining the Polishness of the topology and the continuity of the action, and ensuring that the new topology has the same σ-algebra of Borel sets as the old. Lecture II of Kechris's paper discusses this and other invariant versions of Kuratowski's theorem that have appeared in the literature as well as giving a new application of the method.

Classically it was known that every uncountable Polish space has cardinality $2^{\aleph_0}$. One plausible analogue of this theorem is the still open topological Vaught conjecture, to the effect that whenever a Polish group G acts continuously on a Polish space X either there are only countably many orbits or there are $2^{\aleph_0}$. Alternatively we might be motivated to begin an investigation of the complexity of the quotient spaces arising as the equivalence classes of some Polish group action, and be thus led into the modern theory of definable equivalence relations. These themes occupy the last two lectures of Kechris's paper.

The paper by Foreman is far longer, and in some ways more ambitious. As with Kechris's paper, perhaps one could describe it as a survey, but it contains rather more in the way of hitherto unpublished results.

Foreman's paper comes close to giving a completely self-contained introduction to ergodic theory, but only from a specific point of view: to classify the descriptive set-theoretical complexity of the standard objects of ergodic theory.

Here we understand ergodic theory to be the study of measure-preserving transformations of some *standard Borel probability space*, $(X, \mathcal{B}, \mu)$—that is to say, a Polish space stripped down to its Borel structure, keeping the σ-algebra of Borel sets $\mathcal{B}$, but forgetting the specifics of the original topology, and then giving a measure on the Borel sets with $\mu(X) = 1$. In general the measure-preserving transformation may not itself be Borel, but since we care only about what happens on a measure-one set, and since any measurable function equals a Borel function on a measure-one set, we may restrict our attention to Borel transformations. Alternatively, if we decide to identify all functions that agree almost everywhere, then the invertible measure-preserving functions form a Polish group, sometimes denoted by M_∞. In either case one is close to the traditional objects of descriptive set theory and it is natural that the subject is of interest to set theorists.

After the introduction and survey of the contents, Foreman immediately recalls the definitions of the Borel and projective hierarchies, along with a couple of basic facts, such as the existence of universal sets and the non-reversibility of certain inclusions. The definitions are written to be intelligible to a reader outside logic, though of course it is probably a somewhat bare-bones introduction for the uninitiated.

Then as we go through what might otherwise be a fairly standard introduction to ergodic theory, each object can be given its place in these hierarchies. The assignment of entropy to a measure-preserving transformation is shown to be Borel at exactly the section where it is defined. We come to the definition of K-automorphisms and these are shown to be Borel in the space M_∞. Foreman describes how a measurable transformation can be partly understood in terms of the associated operator on $L^2(X, \mu)$ and then proves that the set of eigenvalues for a unitary operator is uniformly Borel in that operator. The Halmos and von Neumann classification of discrete spectrum operators is seen to work in the Borel context. Some results were known classically: the mixing transformations are $\mathbf{\Pi}^0_2$ (Rohlin); the weakly mixing transformations are $\mathbf{\Pi}^0_3$ (Halmos). Most were not.

The later sections of the paper discuss some joint papers by Beleznay and Foreman and their discovery that the *generalized discrete spectrum* or *measure distal* transformations are not only complete $\mathbf{\Pi}^1_1$ but that certain constructions previously arising in work of Furstenberg give a $\mathbf{\Pi}^1_1$ norm. See the review BSL VII 81. In the paper under review Foreman contents himself with a summary of the main points.

Ultimately, as with the Kechris paper, there is discussion of the theory of definable equivalence relations. Specifically, while it is known that the general isomorphism relation given by M_∞ acting on itself by conjugation is not Borel, and indeed is massively complicated, the isomorphism relation for ergodic measure-preserving transformations is not known to be non-Borel. By way of comparison, the conjugacy equivalence relation on S_∞, the group of all permutations of $\mathbb{N}$, is not only Borel but trivial (count the number of n-cycles for each n) while the corresponding problem for U_∞, the group of unitary operators on Hilbert space, is shown to be Borel by the *spectral theorem* from functional analysis.

Foreman finishes with several open problems. Among other things these bear on placing the complexity of the isomorphism relation of K-automorphisms and generalized discrete spectrum transformations in the known classification schemes for definable equivalence relations.

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