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THE ABC'S OF MICE

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§**1. Introduction.** This is the introductory half of a lecture given at the Association for Symbolic Logic Annual Meeting in Philadelphia in March, 2001. Our goal is to give logicians and advanced students who are unfamiliar with inner model theory a taste of what the subject is about at a level where it is possible to say something meaningful about the intuitions and proofs. At the same time, we wish to avoid overwhelming the reader, so we will leave out many well known closely related theorems and give only short proof sketches. Neither the results nor the proofs found here are due to the author. We will begin near the beginning but still touch on some modern aspects of the theory. The proofs and historical notes that we leave out can be found in the books and articles listed in the references. Our approach to the theory of $0^{\#}$ is a combination of that taken by Dodd in his book and by Steel in his lectures to students. Dodd's introduction is one of the author's favorite essays on inner model theory; another is the introduction to Martin-Steel [6]. In Section 2, we review the theory of $0^{\#}$ in a form that touches on some of the techniques that have been and continue to be generalized. These combine what is commonly called *iteration* and *fine structure*. In Section 3, we give a superficial description of what comes after $0^{\#}$. Those interested in learning about more recent developments in inner model theory are encouraged to see the articles Löwe-Steel [5] and Steel [8], both of which are currently available online at `http://www.math.berkeley.edu/~steel`.

§**2. Baby mice.** Throughout the paper, let Ω denote the class of ordinals. Then $\alpha < \Omega$ means exactly that α is an ordinal. The reader should be familiar with the following definition.

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DEFINITION 2.1 (Gödel).

$$L_0 = \emptyset,$$
$$L_{\alpha+1} = \mathrm{Def}(L_\alpha),$$
$$L_\beta = \bigcup\{\, L_\alpha \mid \alpha < \beta \,\}$$

if β is a limit ordinal, and

$$L = \bigcup\{\, L_\alpha \mid \alpha < \Omega \,\}.$$

For any first order structure $\mathfrak{A}$ with universe A, $\mathrm{Def}(\mathfrak{A})$ denotes the collection of subsets of A that are first order definable with parameters over $\mathfrak{A}$. Above, we are sloughing over the difference between the structure $\langle L_\alpha, \in\rangle$ and its universe L_α in the usual way. We will need to be slightly more careful about this difference beginning in Section 3, but not here.

The following famous theorem involves terms that will need to be explained. (Often, we will state results before giving the definitions.)

THEOREM 2.2 (Gaifman, Solovay, Silver, Kunen). *The following are equivalent.*

(1) *There exists a nontrivial elementary embedding $j\colon L \longrightarrow L$.*
(2) *$\mathcal{M}_0^{\#}$ exists.*
(3) *$0^{\#}$ exists.*
(4) *There exists a closed unbounded class of L-indiscernible ordinals.*

Usually, this theorem is stated with additional clauses that we will not go into. In this section, we will discuss some but not all of the ideas that go into the proof. The proof of the equivalence between (1), (3) and (4) can be found in several textbooks, including those of Jech, Devlin, Dodd and Kanamori. Our primary aim in this section is to motivate and define $\mathcal{M}_0^{\#}$, the minimal active baby mouse. Readers of Dodd's book, which also covers clause (2), will recognize $\mathcal{M}_0^{\#}$ as a kind of *sharp*. In Section 3, we will discuss more powerful mice, hence the use of the term *baby*, which is borrowed from lectures by Steel. Saying what $\mathcal{M}_0^{\#}$ is will take several pages. Our emphasis will be on some of the techniques that have been generalized and are now commonly used in inner model theory.

Let us begin by putting the reading in the shoes of an inner model theorist; in this paragraph, we will describe a certain mind-set that turns out to be fruitful. We began by constructing L level by level. The first ω levels are exactly the hereditarily finite sets, the next ω_1^L levels are exactly the sets that are hereditarily countable in L, and so on. Now we ask ourselves what comes next. For although we have climbed up to the minimal transitive proper class model of ZFC, foundational considerations that fall under the category of *large cardinals* have tempted us to adopt certain theories that extend ZFC. These extensions are not true in L, for they imply that there exists a non-trivial elementary embedding $j\colon L \longrightarrow L$, which is known to fail in L. So how do we continue or revise the construction in a way that buys us the existence of such an embedding? One naïve idea is to continue the construction past all the ordinals and throw in the proper class j at stage Ω or beyond, but this approach leads to some obvious metamathematical problems that we find annoying. Instead, we study j and realize that one consequence of its existence is the existence of a certain canonical hereditarily countable structure $\mathcal{M}_0^\#$, which has the form $\langle L_\lambda, \in, U\rangle$. Moreover, coded into $\mathcal{M}_0^\#$ are L and a non-trivial elementary embedding $i\colon L \longrightarrow L$. It is possible that $i \neq j$, but that is not so important to us here. Our new tactic at stage λ of the construction is to apply the definability operator to $\mathcal{M}_0^\# = \langle L_\lambda, \in, U\rangle$ instead of to $\langle L_\lambda, \in\rangle$. As we march upward from there, the information coded into $\mathcal{M}_0^\#$ is unraveled, giving us a model of ZFC larger than L with the embedding $i\colon L \longrightarrow L$. This accomplishes what the naïve approach described above hoped to and leads to a legitimate interpretation of $\mathcal{M}_0^\#$ as the next canonical object after L.

Now back to the mathematics. Let $j\colon L \longrightarrow L$ be a nontrivial elementary embedding. This means that $j \neq$ identity and for all first order formulas $\varphi(v)$ and all $x \in L$,

$$\varphi^L(x) \longleftrightarrow \varphi^L(j(x)).$$

It is easy to see that:

- $j(\alpha) \geq \alpha$ for all $\alpha < \Omega$, and
- $j(\alpha) > \alpha$ for some $\alpha < \Omega$.

The least $\alpha < \Omega$ such that $j(\alpha) > \alpha$ is called the *critical point* of j and is denoted $\operatorname{crit}(j)$. Let $\kappa = \operatorname{crit}(j)$. Define

$$U_j = \{\, A \subseteq \kappa \mid A \in L \text{ and } \kappa \in j(A)\,\}.$$

LEMMA 2.3. *Suppose that $j\colon L \longrightarrow L$ is an elementary embedding with critical point κ. Then*

(1) *U_j is an ultrafilter on $L \cap \mathcal{P}(\kappa)$,*
(2) *U_j is non-principal,*
(3) *U_j is L-normal, and*
(4) *U_j is L-κ-complete.*

SKETCH. Obviously, either $\kappa \in j(A)$ or $\kappa \notin j(A)$; therefore (1) holds. The main point for (2) is that if a is a bounded subset of κ, then $\kappa \notin a$ and $j(a) = a$. By L-normality, we mean that U_j satisfies Fodor's Lemma, but only for regressive functions that are elements of L. Similarly, L-κ-completeness means that

$$\bigcap_{\alpha<\beta} A_\alpha \in U_j$$

whenever $\beta < \kappa$ and

$$\langle\, A_\alpha \mid \alpha < \beta\,\rangle \in {}^{\beta}(U_j) \cap L.$$

Both (3) and (4) are easy exercises. ⊣

The following commutative diagram is of fundamental importance.

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[dr, "i"'] & L \\ & \mathrm{ult}(L, U_j) \arrow[u, "k"'] \end{tikzcd}$$

Here $\mathrm{ult}(L, U_j)$ is the ultrapower of L by U_j together with an interpretation of the membership relation:

$$\begin{aligned} [f]^L_{U_j} \dot{\in}^{\mathrm{ult}(L,U_j)} [g]^L_{U_j} &\iff \{\, \alpha < \kappa \mid f(\alpha) \in g(\alpha) \,\} \in U_j \\ &\iff j(f)(\kappa) \in j(g)(\kappa). \end{aligned}$$

The maps i and k are defined by the equations

$$i(x) = [\alpha \mapsto x]^L_{U_j}$$

and

$$k([f]^L_{U_j}) = j(f)(\kappa)$$

respectively. Reader beware: this ultrapower differs slightly from the usual kind of model theoretic ultrapower in a few ways, as we make clear in the following remarks.

- A mechanism known as *Scott's Trick* is used to handle the problem of taking the ultrapower of a proper class rather than just a set.
- The functions $f : \kappa \longrightarrow L$ that are used to define the ultrapower are required to be elements of L. This is because U_j only measures subsets of κ that happen to be elements of L. Such ultrapowers are often called Σ_0-*ultrapowers* for the following reasons. By a Σ_0 formula, we mean one in which all of the quantifiers are bounded. The usual proof of Łoś's Theorem from model theory works for Σ_0 formulas when taking Σ_0-ultrapowers. Moreover, the ultrapower map is a Σ_0-elementary embedding, which means that

$$\varphi^{\langle L,\in\rangle}(x) \longleftrightarrow \varphi^{\langle \mathrm{ult}(L,U_j),\dot{\in}^{\mathrm{ult}(L,U_j)}\rangle}(i(x))$$

whenever $\varphi(v)$ is a Σ_0 formula and $x \in L$. (As a reminder, we could have abbreviated "$\varphi^{\langle L,\in\rangle}(x)$" by "$\varphi^L(x)$" above.) However, for the ultrapower of L by U_j, it turns out that Łoś's Theorem holds for all formulas and that

$$i : \langle L, \in\rangle \longrightarrow \langle \mathrm{ult}(L, U_j), \dot{\in}^{\mathrm{ult}(L,U_j)}\rangle$$

is an elementary embedding. The proof uses the fact that L, or rather $\langle L, \in\rangle$, satisfies the Replacement Scheme. *Remark*. We will also take Σ_0-ultrapowers of structures of the form $\langle L, \in, U\rangle$. In such cases, it will not necessarily be true that Łoś's Theorem holds for more than just Σ_0 formulas in the expanded language.

• The definition of k makes sense. Moreover,

$$k\colon \langle \mathrm{ult}(L, U_j), \dot{\in}^{\mathrm{ult}(L,U_j)} \rangle \longrightarrow \langle L, \in \rangle$$

is an elementary embedding.

• From just the fact that there exists a Σ_0-elementary embedding from $\mathrm{ult}(L, U_j)$ to a wellfounded model, we conclude that

$$\langle \mathrm{ult}(L, U_j), \dot{\in}^{\mathrm{ult}(L,U_j)} \rangle$$

is wellfounded.

• Because of this wellfoundedness, by the Mostowski Collapse Theorem, there is a unique isomorphism

$$\pi\colon \langle M, \in \rangle \simeq \langle \mathrm{ult}(L, U_j), \dot{\in}^{\mathrm{ult}(L,U_j)} \rangle$$

with M a transitive class. The custom in set theory is to redefine $\mathrm{ult}(L, U_j)$ to be M, redefine the ultrapower embedding i to be $i \circ \pi^{-1}$, and redefine the embedding k to be $k \circ \pi$. Consider this done, here and throughout. With these new definitions, one can show that

$$\mathrm{ult}(L, U_j) = \{\, i(f)(\kappa) \mid f \in L \,\},$$

and

$$k(i(f)(\kappa)) = j(\kappa).$$

• It is a basic fact due to Gödel that the only transitive proper class model of $\mathsf{ZFC} + \mathsf{V} = \mathsf{L}$ is L itself. Thus $\mathrm{ult}(L, U_j) = L$.

• One can show that $\mathrm{crit}(k) > (\kappa^+)^L$. Hence, by commutativity, $\mathrm{crit}(i) = \kappa$. In particular, i is a non-trivial elementary embedding from L to L with the same critical point as j. One important possible difference between i and j is that i is an ultrapower map. In particular, i is completely determined by U_j, and U_j is a set. As we will see, it is possible to express the existence of an elementary embedding from L to L by axiomatizing the essential properties of the set U_j, rather than by quantifying over proper classes such as j.

• In fact, $U_i = U_j$.

For future reference, we record the following aspects of the above construction.

LEMMA 2.4. *Suppose that* $j\colon L \longrightarrow L$ *is an elementary embedding with critical point* κ. *Let* $X = j[L]$ *and* $Y = \{\, f(\kappa) \mid f \in X \,\}$. *Then* $X \prec Y \prec L$ *and* $Y \simeq \mathrm{ult}(L, U_j) = L$.

Let us continue our study of U_j. We need to recall one of the basic facts that comes out of Gödel's proof that L is a model of GCH, namely that if λ is an infinite cardinal of L, then $L_\lambda = H_\lambda^L$, where

$$H_\lambda = \{\, x \mid x \text{ is hereditarily of cardinality less than } \lambda \,\}.$$

LEMMA 2.5. *Let* $\lambda = (\kappa^+)^L$. *Then*

(1) U_j *is an ultrafilter on* $L_\lambda \cap \mathcal{P}(\kappa)$,
(2) U_j *is* L_λ*-normal,*
(3) U_j *is* L_λ*-*κ*-complete,*
(4) U_j *is* L_λ*-amenable.*

SKETCH. (1)–(3) are immediate from Lemma 2.3 and the fact that $H^L_\lambda = L_\lambda$. By (4), we mean that if $\mathcal{A} \in L_\lambda$, then $U_j \cap \mathcal{A} \in L_\lambda$. Equivalently, if $\alpha < \lambda$, then $U_j \cap L_\alpha \in L_\lambda$. The argument for (4), which is due to Kunen, is as follows. If $\alpha < \lambda$, then $|L_\alpha|^L \leq \kappa$. Let $\langle A_i \mid \eta < \kappa \rangle$ be an enumeration in L of $\mathcal{P}(\kappa) \cap L_\alpha$. Say

$$j(\langle A_\eta \mid \eta < \kappa \rangle) = \langle B_\eta \mid \eta < j(\kappa) \rangle.$$

Then

$$U_j = \{ B_\eta \cap \kappa \mid \eta < \kappa \text{ and } \kappa \in B_\eta \}. \qquad \dashv$$

Active baby premice, defined below, are structures with some of the basic properties of $\langle L_\lambda, \in, U_j \rangle$.

DEFINITION 2.6. A structure $\mathcal{M}$ is an *active baby premouse* if and only if for some $\kappa < \lambda$ and U,

(1) $\mathcal{M} = \langle L_\lambda, \in, U \rangle$,
(2) $\langle L_\lambda, \in \rangle \models \mathsf{ZF} - \mathsf{Power\ Set\ Axiom}$,
(3) $\mathcal{M} \models$ "κ is the largest cardinal",
(4) $\mathcal{M} \models$ "U is a κ-complete non-principal normal ultrafilter over κ",
(5) $\mathcal{M}$ is amenable, and
(6) $\lambda = (\kappa^+)^{\mathrm{ult}(L_\lambda, U)}$.

DEFINITION 2.7. Suppose that $j : L \longrightarrow L$ is an elementary embedding with critical point κ. Let $\lambda = (\kappa^+)^L$. Then $\langle L_\lambda, \in, U_j \rangle$ is the active baby premouse *derived from* j.

Regarding Definition 2.6, we have left the meaning of clause 6) somewhat vague in the case in which $\mathrm{ult}(L_\lambda, U)$ is illfounded; for the careful reader, it is implicitly assumed that $\mathrm{ult}(L_\lambda, U)$ is wellfounded up to λ. Notice that the entire structure of an active baby premouse $\langle L_\lambda, \in, U \rangle$ is determined by U. *Passive baby premice* are merely structures of the form $\langle L_\lambda, \in, \emptyset \rangle$. What turns baby premice into baby mice is the form of iterability that we are about to describe. Iterability is of central importance in inner model theory; the reasons are discussed just before Lemma 2.20.

Towards the definition of iterability, consider an arbitrary active baby premouse $\mathcal{M}_0 = \langle L_{\lambda_0}, \in, U_0 \rangle$. Assume that the ultrapower of L_{λ_0} by U_0 is wellfounded. Let

$$L_{\lambda_1} = \mathrm{ult}(L_{\lambda_0}, U_0)$$

and

$$i_{0,1}\colon L_{\lambda_0} \longrightarrow L_{\lambda_1}$$

be the ultrapower embedding. Then $i_{0,1}$ is an elementary embedding from L_{λ_0} to L_{λ_1}. (The proof uses Łoś's Theorem for Σ_0 formulas and the fact that L_{λ_0} satisfies the Replacement Scheme.) Some straightforward calculations show that

$$\kappa_0 = \operatorname{crit}(i_{0,1}),$$
$$i_{0,1}(\kappa_0) > \lambda_0,$$

and

$$\lambda_1 = \sup(i_{0,1}[\lambda_0]).$$

Let

$$\kappa_1 = j(\kappa_0),$$
$$U_1 = \bigcup_{\alpha<\lambda_0} i_{0,1}(U_0 \cap L_\alpha)$$

and

$$\mathcal{M}_1 = \langle L_{\lambda_1}, \in, U_1\rangle.$$

Note that amenability is used to make sense of the definition of U_1. Since $\mathcal{M}_0$ does not satisfy the Replacement Scheme for formulas in the expanded language, we initially only know that $i_{0,1}$ is a Σ_0-elementary embedding from $\mathcal{M}_0$ to $\mathcal{M}_1$. Because of the limited elementarity of $i_{0,1}$, some work is needed to see that $\mathcal{M}_1$ is a premouse. The calculations are traditionally organized as follows. First show that $i_{0,1}$ is a *cofinal* embedding in the sense that

$$\lambda_1 = \sup(i_{0,1}[\lambda_0]).$$

Second show that since $i_{0,1}\colon \mathcal{M}_0 \longrightarrow \mathcal{M}_1$ is cofinal and Σ_0-elementary, it is Σ_1-elementary. That is, show that $i_{0,1}$ is elementary with respect to formulas of the form $\exists w\, \varphi(v, w)$ where φ is a Σ_0 formula. Finally, show that premousehood is preserved upward by cofinal Σ_1-elementary embeddings.

The definition of $\mathcal{M}_1$ was the first step of a transfinite recursive definition. Given $\mathcal{M}_\alpha$, assuming that it is an active baby premouse, we define

$$\mathcal{M}_{\alpha+1} = \langle L_{\lambda_{\alpha+1}}, \in, U_{\alpha+1}\rangle$$

and

$$i_{\alpha,\alpha+1}\colon \mathcal{M}_\alpha \longrightarrow \mathcal{M}_{\alpha+1}$$

similarly. If β is a limit ordinal, then $\mathcal{M}_\beta$ is the direct limit of $\mathcal{M}_\alpha$ for $\alpha < \beta$ under the obvious maps.

Definition 2.8. An active baby premouse $\mathcal{M}$ is *γ-iterable* if and only if setting $\mathcal{M}_0 = \mathcal{M}$ in the construction above, for all $\beta < \gamma$,

- if $\beta = \alpha + 1$, then $\mathrm{ult}(L_{\lambda_\alpha}, U_\alpha)$ is wellfounded, and
- if β is a limit ordinal, then the direct limit of $\mathcal{M}_\alpha$ for $\alpha < \beta$ is wellfounded.

DEFINITION 2.9. $\mathcal{M}$ is an *active baby mouse* if and only if $\mathcal{M}$ is an Ω-iterable baby premouse.

LEMMA 2.10. *If j is a non-trivial elementary embedding from L to L and $\mathcal{M}$ is the active baby premouse derived from j, then $\mathcal{M}$ is Ω-iterable.*

BARE BONES SKETCH. Let $U_0 = U_j$ and $i_{0,1} : L \longrightarrow \mathrm{ult}(L, U_0)$ be the ultrapower map. We explained earlier that $\mathrm{ult}(L, U_0)$ is wellfounded because it is isomorphic to an elementary substructure of L. (See the discussion leading up to Lemma 2.4; the way we stated Lemma 2.4 gives some indication of what we are about to do.) This is enough to conclude that $\mathcal{M}$ is 2-iterable. To prove Lemma 2.10, one proves a fact that is stronger than what is needed, namely that $\langle L, \in, U_0 \rangle$ is γ-iterable for every $\gamma < \Omega$, where this sort of iterability is given the obvious meaning. The proof is by induction on γ. With some work, the case $\gamma = \alpha + 2$ can be seen as tracing back to the case $\gamma = 2$, which we have already handled. So let us think about the case in which γ is a limit ordinal. Let $i_{\alpha,\beta}$ be the iteration map from the αth model, which is L, to the βth model, which is also L, whenever $\alpha < \beta < \gamma$.

For an arbitrary continuous chain $\langle\, X_\alpha \mid \alpha < \gamma \,\rangle$ of proper class elementary substructures of L, we can form the following diagram by taking Mostowski collapses.

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{id}"'] & \cdots \arrow[r, "\mathrm{id}"'] & X_\alpha \arrow[r, "\mathrm{id}"'] & X_{\alpha+1} \arrow[r, "\mathrm{id}"'] & \cdots \arrow[r, "\mathrm{id}"'] & \bigcup_{\beta<\gamma} X_\beta \\
L \arrow[u, "\pi_0"'] \arrow[r] & \cdots \arrow[r] & L \arrow[u, "\pi_\alpha"'] \arrow[r, "\sigma_{\alpha,\alpha+1}"'] & L \arrow[u, "\pi_{\alpha+1}"'] \arrow[r] & \cdots \arrow[r] & \lim(\langle L \rangle_{\beta<\gamma}, \langle \sigma_{\beta',\beta} \rangle_{\beta'<\beta<\gamma}) \arrow[u]
\end{tikzcd}$$

Notice that the direct limit along the bottom is wellfounded since it is isomorphic to the union along the top. (So the direct limit along the bottom is isomorphic to L.) A more compressed way to draw the above commutative diagram is

$$\begin{tikzcd}
L & & & & & \\
L \arrow[u, "\pi_0"] \arrow[r] & \cdots \arrow[r] & L \arrow[ull, "\pi_\alpha"'] \arrow[r, "\sigma_{\alpha,\alpha+1}"'] & L \arrow[ulll, "\pi_{\alpha+1}"'] \arrow[r] & \cdots \arrow[r] & \lim(\langle L \rangle_{\beta<\gamma}, \langle \sigma_{\beta',\beta} \rangle_{\beta'<\beta<\gamma}) \arrow[ulllll]
\end{tikzcd}$$

where $X_\alpha = \pi_\alpha[L]$. Let us say that $\langle\, X_\alpha \mid \alpha < \gamma \,\rangle$ is a *realization* if and only if the embeddings along the bottom of the diagrams are exactly the iteration maps, that is

$$\sigma_{\alpha,\beta} = i_{\alpha,\beta}$$

for $\alpha < \beta < \gamma$. If there is a realization, then the γth model of the iteration is also wellfounded, as desired.

Let $X_0 = \{ x \in L \mid i_{\alpha,\beta}(x) = x$ whenever $\alpha < \beta < \gamma \}$. Using the definable wellorder of L, one can show that $X_0 \prec L$. One can also show that since each $i_{\alpha,\beta}$ comes from an iterated ultrapower construction, X_0 is a proper class. For $\alpha < \gamma$, let $X_{\alpha+1} = \{ f(\kappa_\alpha) \mid f \in X_\alpha \}$. For limit ordinals $\beta < \gamma$, let $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$. With these definitions and some work, things work out as desired. We refer the reader to Chapter 12 of Dodd's book for the details. ⊣

An important reason we wish to iterate mice is that the critical points are indiscernible in L.

Lemma 2.11. *Let $\mathcal{M}$ be an active baby mouse and $\langle \kappa_\alpha \mid \alpha < \Omega \rangle$ be the increasing sequence of critical points in the iteration of $\mathcal{M}$. Then*

- $\{ \kappa_\alpha \mid \alpha < \Omega \}$ *is a closed unbounded class, and*
- *if $\varphi(v_0, \ldots, v_{n-1})$ is a formula in the language of* ZFC *and*

$$\alpha_0 < \cdots < \alpha_{n-1},$$

then

$$\varphi^L(\kappa_{\alpha_0}, \ldots, \kappa_{\alpha_{n-1}}) \longleftrightarrow \varphi^L(\kappa_0, \ldots, \kappa_{n-1}).$$

Sketch. The fact that the class of critical points is closed and unbounded follows easily from the definition of the iteration.

To prove the claim about L-indiscernibility, first apply the Reflection Theorem to get $\alpha_n > \alpha_{n-1}$ that has the absoluteness property

$$\varphi^L(\kappa_{\alpha_0}, \ldots, \kappa_{\alpha_{n-1}}) \longleftrightarrow \varphi^{L_{\kappa_{\alpha_n}}}(\kappa_{\alpha_0}, \ldots, \kappa_{\alpha_{n-1}}).$$

Then show that the righthand-side holds if and only if $\varphi(\xi_0, \ldots, \xi_{n-1})$ holds in L_{ξ_n} for $(U \times \cdots \times U)$-a. e. sequence $\xi_n > \cdots > \xi_0$ from κ. The details are left to the reader. ⊣

Lemma 2.12. *If there is an active baby mouse, then there is a non-trivial elementary embedding from L to L.*

Sketch. From Lemma 2.11, it follows that κ_0 is not definable in L from the other critical points. Let X be the the proper class of all elements of L that are definable from parameters in $\kappa_1 < \cdots < \kappa_\alpha < \cdots$ and let $\pi\colon L \simeq X$ be the isomorphism given by the Mostowski Collapse Theorem. Then π is a non-trivial elementary embedding from L to L. ⊣

Under the hypothesis that there exists a closed unbounded class $C = \{c_0 < \cdots < c_\alpha < \cdots\}$ of L-indiscernibles, Solovay defined $0^\#$ to be the set of Gödel numbers $\ulcorner\varphi\urcorner$ of first order formulas $\varphi(v_0, \ldots, v_{n-1})$ such that $\varphi(c_0, \ldots, c_{n-1})$ holds in L. It turns out that Solovay's definition does not depend on C, hence

$$0^\# = \{ \ulcorner\varphi\urcorner \mid \varphi^L(\kappa_0, \ldots, \kappa_{n-1}) \}$$

under the hypotheses of Lemma 2.11. Notice that $0^\#$ is countable since it is a set of natural numbers. The above characterization of $0^\#$ in terms of mice has the recursion theoretic corollary that $0^{\#} \leq_T \mathcal{M}$ whenever $\mathcal{M}$ is a countable active baby mouse. Below, we will define a particular countable active baby mouse $\mathcal{M}_0^{\#}$ with the property that

$$\mathcal{M}_0^{\#} \equiv_T 0^{\#}.$$

Solovay showed that $\{0^{\#}\}$ is a Π^1_2-singleton. We will see directly that that $\{\mathcal{M}_0^{\#}\}$ is Π^1_2-definable in the codes for hereditarily countable sets.

LEMMA 2.13. *If $\mathcal{M} = \langle L_\lambda, \in, U\rangle$ is a $|\lambda|^+$-iterable active baby premouse, then $\mathcal{M}$ is an active baby mouse.*

SKETCH. Suppose that $\mathcal{M}$ is not γ-iterable. Let $\theta > \gamma$ be regular. Let $X \prec H_\theta$ with

$$\lambda \cup \{\mathcal{M}, \gamma\} \subseteq X$$

and $|X| = |\lambda|$. Say $\pi \colon N \simeq X$ with N transitive.

$$N \models \text{“}\pi^{-1}(\mathcal{M}) \text{ is not } \pi^{-1}(\gamma)\text{-iterable.”}$$

Because $\pi^{-1}(\mathcal{M}) = \mathcal{M}$ and Σ_1-formulas are absolute from N to V, $\mathcal{M}$ is not $\pi^{-1}(\gamma)$-iterable. Since $\pi^{-1}(\gamma) < |\lambda|^+$, $\mathcal{M}$ is not $|\lambda|^+$-iterable. $\dashv$

COROLLARY 2.14. *A countable baby premouse $\mathcal{M}$ is a mouse if and only if $\mathcal{M}$ is ω_1-iterable.*

PROOF. Apply Lemma 2.13 with $\lambda = \omega_1$. $\dashv$

COROLLARY 2.15. *The set*

$$\{\, x \subseteq \omega \mid x \textit{ codes a countable baby mouse} \,\}$$

is Π^1_2-definable.

PROOF. Let $\mathcal{M}$ be a countable structure and $x \subseteq \omega$ be a code for $\mathcal{M}$. By Definition 2.6, $\mathcal{M}$ is a premouse if and only if $\mathcal{M}$ satisfies certain first order conditions. Thus $\mathcal{M}$ is a premouse if and only if a certain bounded formula holds about $\mathcal{M}$ if and only if a certain $\Delta^1_1(x)$ property holds. By Corollary 2.14, the iterability of $\mathcal{M}$ is Π_1 over the structure $\langle HC, \in\rangle$, hence $\Pi^1_2(x)$. $\dashv$

DEFINITION 2.16. Let $\mathcal{M} = \langle L_\lambda, \in, U\rangle$ be a baby premouse.

- Given a formula $\varphi(u, v)$ and $a \in L_\lambda$, if there exists $b \in L_\lambda$ such that

  $$\mathcal{M} \models \varphi(a, b),$$

  then let $\tau^{\mathcal{M}}_\varphi(a)$ be the $<_L$-least such b. Otherwise, leave $\tau^{\mathcal{M}}_\varphi(a)$ undefined.
- If $X \subseteq L_\lambda$, then let

  $$\mathrm{Hull}^{\mathcal{M}}_1(X) = \{\, \tau^{\mathcal{M}}_\varphi(a) \mid \varphi \text{ is } \Sigma_1 \text{ and } a \in X \,\}.$$

- $\mathcal{M}$ is *minimal* if and only if $\mathrm{Hull}^{\mathcal{M}}_1(\emptyset) = L_\lambda$.

We remark that Jensen's *fine structure theory* begins with the theorem that the levels of L are sound, which is a property closely related to minimality. For the reader who happens to be familiar with Jensen's theory: minimality says that ω is the Σ_1 projectum of $\mathcal{M}$, the Σ_1 standard parameter of $\mathcal{M}$ is empty, and $\mathcal{M}$ is Σ_1 sound.

Clearly, if $\mathcal{M}$ is a minimal baby premouse, then $\mathcal{M}$ is a countable structure. Thus a minimal baby premouse is a mouse if and only if it is ω_1-iterable.

DEFINITION 2.17. If it exists, let $\mathcal{M}_0^\#$ be the unique minimal active baby mouse.

COROLLARY 2.18. *If $\mathcal{M}_0^\#$ exists and is coded by $x \subseteq \omega$, then $\{x\}$ is Π^1_2-definable.*

LEMMA 2.19. *If there exists an active baby mouse, then there exists a minimal active baby mouse.*

SKETCH. Let $\mathcal{M}$ be an arbitrary active baby mouse. Let $\mathcal{M}^*$ be the structure whose universe is the transitive collapse of $\mathrm{Hull}_1^{\mathcal{M}}(\emptyset)$. The map $\pi\colon \mathcal{M}^* \longrightarrow \mathcal{M}$ is Σ_1-elementary, which is enough to conclude that $\mathcal{M}^*$ is an active baby premouse. Minimality holds by design. The remaining clause that needs to be verified is iterability. The idea is to define so-called *copy* maps π_α by recursion on $\alpha < \Omega$ so that the following diagram commutes.

$$
\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "i_{0,1}"] & \mathcal{M}_1 \arrow[r] & \cdots \arrow[r] & \mathcal{M}_\alpha \arrow[r] & \cdots \\
\mathcal{M}_0^* \arrow[u, "\pi_0"] \arrow[r, "i^*_{0,1}"'] & \mathcal{M}_1^* \arrow[u, "\pi_1"] \arrow[r] & \cdots \arrow[r] & \mathcal{M}_\alpha^* \arrow[u, "\pi_\alpha"] \arrow[r] & \cdots
\end{tikzcd}
$$

Along the bottom, we have the iteration of $\mathcal{M}^*$ and, along the top, that of $\mathcal{M}$. We set $\pi_0 = \pi$. The definition at successor steps is

$$\pi_{\alpha+1}\big([f]^{\mathcal{M}_\alpha^*}_{U_\alpha^*}\big) = [\pi_\alpha(f)]^{\mathcal{M}_\alpha}_{U_\alpha}.$$

If α is a limit ordinal, then use the definition of $\mathcal{M}_\alpha^*$ and $\mathcal{M}_\alpha$ in terms of direct limits to define π_α in the obvious way. We maintain inductively that $\pi_\alpha\colon \mathcal{M}_\alpha^* \longrightarrow \mathcal{M}_\alpha$ is a Σ_1-elementary embedding. By the iterability of $\mathcal{M}$, every $\mathcal{M}_\alpha$ is wellfounded. Hence, every $\mathcal{M}_\alpha^*$ is wellfounded. ⊣

The proof of the next lemma uses a so-called *comparison* argument, which is also called *co-iteration*. Of all the techniques illustrated in this paper, we consider this to be the most important in that nearly every theorem in modern inner model theory has some kind of co-iteration at its heart. In the following lemma, we will see how iterability makes the objects we are dealing with canonical and definable.

LEMMA 2.20. *There is at most one minimal active baby mouse.*

SKETCH. Let $\mathcal{M}$ and $\mathcal{N}$ be minimal active baby mice. Denote the iteration maps as follows.

$$\mathcal{M}_0 \xrightarrow{i_{0,1}} \mathcal{M}_1 \longrightarrow \cdots \longrightarrow \mathcal{M}_\alpha \longrightarrow \cdots$$

$$\mathcal{N}_0 \xrightarrow[j_{0,1}]{} \mathcal{N}_1 \longrightarrow \cdots \longrightarrow \mathcal{N}_\alpha \longrightarrow \cdots$$

We claim that there exist an ordinal $\delta < \omega_1$ such that $\mathcal{M}_\delta = \mathcal{N}_\delta$. Modulo this claim, we have

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{i_{0,\delta}} & \mathcal{M}_\delta \\ & & \| \\ \mathcal{N} & \xrightarrow[j_{0,\delta}]{} & \mathcal{N}_\delta \end{array}$$

Since $\mathcal{M}$ is minimal, $\mathcal{M}$ is determined by a set of Σ_1-sentences without parameters that hold in $\mathcal{M}$. Similarly for $\mathcal{N}$. Since $i_{0,\delta}$ and $j_{0,\delta}$ are Σ_1-elementary, the two Σ_1-theories are the same, so $\mathcal{M} = \mathcal{N}$.

Now we turn to the proof of the claim. Let $\theta = \aleph_2$ and $X \prec H_\theta$ be such that $\mathcal{M}, \mathcal{N} \in X$ and X is countable. Say $\pi\colon P \simeq X$ with P transitive and $\delta = \operatorname{crit}(\pi)$. Then $\pi(\delta) = \omega_1$ and $\delta = \omega_1^P$. Let $\langle\, \kappa_\alpha \mid \alpha < \omega_1 \,\rangle$ be the sequence of critical points for the iteration of $\mathcal{M}$ and let $\langle\, U_\alpha \mid \alpha < \omega_1 \,\rangle$ be the corresponding filters. One can easily verify that

$$\kappa_{\omega_1} = \omega_1,$$

$$\kappa_\delta = \delta$$

and

$$i_{\delta,\omega_1}(\kappa_\delta) = \kappa_{\omega_1}.$$

The definition of the iteration of $\mathcal{M}$ is sufficiently absolute that for $\alpha < \Omega \cap P$,

$$(\mathcal{M}_\alpha)^P = \mathcal{M}_\alpha.$$

In particular, $\mathcal{M}_\alpha \in P$ for $\alpha \leq \delta$ and, since these structures are hereditarily countable in P, $\pi(\mathcal{M}_\alpha) = \mathcal{M}_\alpha$. On the other hand,

$$\pi(\mathcal{M}_\delta) = \mathcal{M}_{\omega_1}.$$

Analogous facts hold for the iteration of $\mathcal{N}$.

To prove the claim, we will show that $\mathcal{M}_\delta = \mathcal{N}_\delta$. The first step is to show that i_{δ,ω_1} equals the restriction of π to the universe of $\mathcal{M}_\delta$. Consider an arbitrary element A of $\mathcal{M}_\delta$. Since δ is a limit ordinal, $\mathcal{M}_\delta$ is the direct limit of $\mathcal{M}_\alpha$ for $\alpha < \delta$. So there is an ordinal $\alpha < \delta$ and an element a of $\mathcal{M}_\alpha$ such that $i_{\alpha,\delta}(a) = A$. Then $\pi(a) = a$ and

$$\pi(A) = \pi(i_{\alpha,\delta}(a)) = \pi(i_{\alpha,\delta})(\pi(a)) = i_{\alpha,\omega_1}(a) = i_{\delta,\omega_1}(i_{\alpha,\delta}(a)) = i_{\delta,\omega_1}(A).$$

Therefore

$$A \cap \kappa_\delta \in U_\delta \iff \kappa_\delta \in i_{\delta,\delta+1}(A) \iff \kappa_\delta \in i_{\delta,\omega_1}(A) \iff \delta \in \pi(A).$$

This characterizes U_δ in terms of π, and hence also $\mathcal{M}_\delta$ in terms of π. By symmetry, exactly the same characterization holds for $\mathcal{N}_\delta$. Hence, $\mathcal{M}_\delta = \mathcal{N}_\delta$, as desired. (It is worth adding that, because $\mathcal{M}_\delta$ and $\mathcal{N}_\delta$ are active baby mice with the same critical point, δ, they have the same ordinal height, namely $(\delta^+)^L$, and therefore the same power set of δ. To see this, consider clause 6 in the definition of *active baby premouse* and keep iterating.) $\dashv$

Let us summarize where we stand with regard to Theorem 2.2. At some level, we have discussed (2) $\Longrightarrow$ (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2) and, also, (2) $\Longrightarrow$ (3), and supplied a few of the details. Even though there is much more to the theory of baby mice, it is time to move on.

§3. Beyond baby mice. In this section, we will give the reader an idea of what non-baby mice look like. The particular form of the models that we are about to describe is the work of Jensen, Dodd, Mitchell, Baldwin, Martin, and Steel.

We will begin by making sense of

$$L,\ \mathcal{M}_0^{\#},\ L(0^{\#}),\ \mathcal{M}_0^{\#\#},\ L(0^{\#\#}),\ \ldots$$

in terms of a natural progression of models about which researchers have gained significant understanding up to and beyond the level of Woodin cardinals, but not yet to the level of supercompact cardinals. The first step is to see that each of these models is constructed from a so-called *coherent sequence of extenders*. Extenders will be defined in this section; the reader can simply substitute the word *filter* until then. The goal of such a uniform description is to understand the progression of models more clearly and to see how it might be possible to climb arbitrarily far up the large cardinal hierarchy. The point of coherence is to organize the extenders in increasing order without leaving any gaps in the sense of the Mitchell order on extenders. This goal and method of attack are among Mitchell's great insights.

There are some slightly technical considerations in Subsection 3.1, but it helps to get one's hands dirty at the start. One can make do with ultrafilters rather than extenders up to and somewhat beyond the level of measurable cardinals, so extenders are not discussed until Subsection 3.3. Coherence is discussed in Subsection 3.2

3.1. Mice and iterated sharps. For any transitive set X, we define $L(X)$ to be the minimal transitive class model of ZF that has X as an element. Say $\mathcal{M}_0^{\#} = \langle L_\lambda, \in, U\rangle$. The fact that $\mathcal{M}_0^{\#} \equiv_T 0^{\#}$ implies that $L(0^{\#}) = L(\mathcal{M}_0^{\#})$. Here is a useful description of $L(0^{\#})$. First, construct L_λ as usual. Second, apply the definability operator to $\mathcal{M}_0^{\#}$. This means that the $(\lambda + 1)$-st level consists of all subsets of L_λ that are definable over $\mathcal{M}_0^{\#}$. It is important to note that there are no new bounded subsets of λ definable over L_λ, since λ is a cardinal in L, whereas over $\mathcal{M}_0^{\#}$ there are new definable subsets of ω, for example, $0^{\#}$. Finally, construct the remaining levels by applying the definability operator at successor stages and by taking unions at limit stages.

Now we will make this construction more rigorous and possibly a bit too general. Let us suppose that we are given a class sequence

$$E = \langle\, E_\alpha \mid \alpha < \Omega\,\rangle.$$

In the description of $L(0^{\#})$ above, what we would have is $E_\lambda = U$ and $E_\alpha = \emptyset$ for all ordinals $\alpha \neq \lambda$. We will define a structure

$$\mathcal{L}_\alpha[E] = \langle L_\alpha[E], \in, E{\restriction}\alpha, E_\alpha\rangle$$

for $\alpha < \Omega$. For $\mathcal{L}_\alpha[E]$ to be a bonafide first order structure, we must assume that $E_\alpha \subseteq L_\alpha[E]$ for all $\alpha < \Omega$. Because all transitive models of set theory have the same hereditarily finite sets, it makes sense to begin our construction at stage ω. The recursive definition is

$$L_\omega[E] = \mathrm{HF},$$
$$L_{\alpha+1}[E] = \mathrm{Def}(\mathcal{L}_\alpha[E]),$$

and

$$L_\beta[E] = \bigcup\{\, L_\alpha[E] \mid \alpha < \beta\,\}$$

if β is a limit ordinal. Finally, set

$$L[E] = \bigcup\{\, L_\alpha[E] \mid \alpha < \Omega\,\}$$

and

$$\mathcal{L}[E] = \langle L[E], \in, E\rangle.$$

We call $\mathcal{L}_\alpha[E]$ *passive* if $E_\alpha = \emptyset$ and *active* otherwise.

For the rest of this subsection, we assume that $L(0^{\#})$ is laid out as a model of the form $\mathcal{L}[E]$ in which $E_\lambda = U$ is the L-ultrafilter with critical point κ that comes from $\mathcal{M}_0^{\#}$, and $E_\alpha = \emptyset$ for all $\alpha \neq \lambda$. In particular, the only active level of $\mathcal{L}[E]$ is $\mathcal{L}_\lambda[E]$.

It would be convenient to be able to say that $\mathcal{M}_0^{\#} = \mathcal{L}_\lambda[E]$, but this is not true for the boring reason that

$$\mathcal{M}_0^{\#} = \langle L_\lambda, \in, U\rangle$$

whereas

$$\mathcal{L}_\lambda[E] = \langle L_\lambda, \in, \langle \emptyset, \dots, \emptyset, \dots\rangle, U\rangle.$$

Let us consider all of the definitions given in Section 2 to be changed in the obvious way so that we may legitimately write $\mathcal{M}_0^{\#} = \mathcal{L}_\lambda[E]$.

Thus $\mathcal{L}_\lambda[E]$ is now the unique minimal active baby mouse. Originally, we considered every level of L to be a passive baby mouse. With the new changes, we can say that $\mathcal{L}_\alpha[E]$ is a passive baby mouse for all $\alpha < \lambda$. The higher levels of $\mathcal{L}[E]$ are also considered passive premice according to the intended change in definitions. What turns premice into mice is still a form of iterability. In thinking about what iterability should mean for levels $\mathcal{L}_\alpha[E]$ with $\alpha > \lambda$, consider the fact that $\kappa = \mathrm{crit}(U)$ is a countable ordinal in $L_{\lambda+1}[E]$. In particular, U is not an $L_\alpha[E]$-ultrafilter if $\lambda < \alpha$, so it does not make sense to apply U to $\mathcal{L}_\alpha[E]$. Consequently, we let the ultrapower of $\mathcal{L}[E]$ by U be $\mathrm{ult}(\mathcal{L}_\lambda[E], U)$. In other words, at the first step of the iteration of $\mathcal{L}[E]$, we *drop* to $\mathcal{L}_\lambda[E] = \mathcal{M}_0^\#$ and iterate from there. Since $\mathcal{M}_0^\#$ is iterable by definition, every level of $\mathcal{L}[E]$ is a mouse, including $\mathcal{L}[E]$ itself.

Next we describe $\mathcal{M}_0^{\#\#}$. We will have to generalize further the terms introduced in Section 2. Still assuming that $L(0^\#) = L[E]$ as above, suppose that $\widetilde{j}\colon L[E] \longrightarrow L[E]$ is a nontrivial elementary embedding. Say

$$\widetilde{\kappa} = \mathrm{crit}(\widetilde{j}),$$
$$\widetilde{U} = \{\, A \subseteq \widetilde{\kappa} \mid A \in L[E] \text{ and } \widetilde{\kappa} \in \widetilde{j}(A) \,\},$$

and

$$\widetilde{\lambda} = (\widetilde{\kappa}^+)^{L[E]}.$$

It is not hard to see that λ is a countable ordinal in $L[E]$, while $\widetilde{\kappa}$ is an inaccessible cardinal in $L[E]$. In particular,

$$\kappa < \lambda < \omega_1^{L[E]} < \widetilde{\kappa} < \widetilde{\lambda}.$$

Define a new sequence F such that

$$F \restriction \widetilde{\lambda} = E \restriction \widetilde{\lambda}$$

and

$$F_{\widetilde{\lambda}} = \widetilde{U}.$$

Let $\mathcal{M} = \mathcal{L}_{\widetilde{\lambda}}[F]$. Then $\mathcal{M}$ should also be considered a premouse.

One kind of iteration of $\mathcal{M}$ consists only of applications of $\widetilde{U}$ and its images. A second kind of iteration drops immediately to $\mathcal{L}_\lambda[F] = \mathcal{M}_0^\#$ and consists only of applications U and its images. There is also a third possibility that is basically the first followed by the second. (Notice that since U is countable in $\mathcal{M}$, U is fixed in iterations by $\widetilde{U}$ and the images of $\widetilde{U}$.) Because $\widetilde{U}$ is derived from an elementary embedding from $L[E]$ to $L[E]$, as in the proof of Lemma 2.10, one can show that $L[E]$ is iterable by $\widetilde{U}$, hence the first kind of iteration never reaches an illfounded model. Since $\mathcal{M}_0^\#$ is iterable by definition, the second kind of iteration is also well-behaved. Therefore $\mathcal{M}$ is iterable in the sense involving drops described above, which is to say that $\mathcal{M}$ is a mouse in the new sense.

Now $\mathcal{M}_0^{\#\#}$ is the structure whose universe is the transitive collapse of

$$\mathrm{Hull}_1^{\mathcal{M}}(\emptyset)$$

and whose predicates are determined accordingly. The point of taking a Skolem hull is to ensure an appropriate form of minimality; see Lemma 2.19. The theory of elementary embeddings from $L(0^{\#})$ to $L(0^{\#})$, $0^{\#\#}$, and $\mathcal{M}_0^{\#\#}$ runs parallel to that described in Section 2.

Continuing in this way, one defines mice $\mathcal{M}_0^{\#\cdots\#}$ corresponding to $0^{\#\cdots\#}$. An example of a mouse beyond these is the minimal transitive proper class model of ZFC that is closed under the operation $X \mapsto X^{\#}$. This model too has a sharp. There are mice above all of these on the way up to a measurable cardinal.

3.2. Measurable cardinals. Abandon the meanings of "U" and "E" from the previous subsection. Let us take a big jump in the mouse hierarchy and give a highly incomplete account of what the minimal transitive proper class model with a measurable cardinal looks like in terms of mice. This model was studied by Solovay, Silver and Kunen using a different description, which we do not go into. In order for this model to be the universe of a mouse $\mathcal{L}[E]$, the definition of *mouse* must again be generalized; let us describe some of the new requirements.

As one would expect from looking at Definition 2.6, the premousehood of $\mathcal{L}_\lambda[E]$ for $\lambda < \Omega$ requires, among other things, that either $E_\lambda = \emptyset$, or else for some $\kappa < \lambda$,

(1) $\langle L_\lambda[E], \in\rangle \models \mathsf{ZF} - \mathsf{Power\ Set\ Axiom}$,
(2) $\mathcal{L}_\lambda[E] \models$ "κ is the largest cardinal",
(3) $\mathcal{L}_\lambda[E] \models$ "E_λ is a κ-complete non-principle normal ultrafilter over κ",
(4) $\mathcal{L}_\lambda[E]$ is amenable, and
(5) $\lambda = (\kappa^+)^{\mathrm{ult}(L_\lambda[E], E_\lambda)}$.

There is an important additional condition known as *coherence*, which says that for every $\lambda < \Omega$, if $E_\lambda \neq \emptyset$ and by taking an ultrapower we obtain the structure

$$\mathrm{ult}(\mathcal{L}_\lambda, E_\lambda) = \mathcal{L}_{\lambda^*}[F],$$

then

$$F \restriction \lambda = E \restriction \lambda$$

and

$$F_\lambda = \emptyset.$$

By the first clause, for every $\alpha < \lambda$, E_α is an element of the ultrapower by E_λ, which is to say that each E_α is less than E_λ in terms of the so-called Mitchell order. The second clause in the definition of coherence basically says that among all the filters vying for index λ, E_λ has minimal Mitchell order. It should be noted that the sequences E that were described in Subsection 3.1 are coherent for trivial reasons.

In the case of the minimal transitive proper class model with a measurable cardinal, E happens to be tremendously complicated. Say μ is the unique measurable cardinal of $\mathcal{L}[E]$ and $\mathcal{L}[E]$ satisfies that U is the normal measure over μ. Let $\nu = (\mu^+)^{L[E]}$. Then it turns out that

- $E_\alpha = \emptyset$ for every $\alpha > \nu$,
- $E_\nu = U$, and
- for every cardinal $\mu' \leq \mu$, the collection of $\alpha < \mu'$ such that $E_\alpha \neq \emptyset$ is unbounded in μ'.

Other notions from Section 2 need to be broadened. For example, since not every active level must be countable, a different notion of minimality is necessary; the right definition involves Jensen's fine structure, which we do not get into here.

Drops, hence iterability, are also slightly different, since we must deal with the possibility that for some $\alpha < \beta < \Omega$, E_α is an $L_\beta[E]$-ultrafilter. The basic idea is to let γ be the greatest $\beta < \Omega$ such that E_α is an $L_\beta[E]$-ultrafilter, and to define $\mathrm{ult}(\mathcal{L}[E], E_\alpha)$ to be $\mathrm{ult}(\mathcal{L}_\gamma[E], E_\alpha)$.

3.3. Woodin cardinals. Finally, we give a brief description of what lies beyond measurable cardinals. More specifically, this section is about the models constructed in Mitchell-Steel [7] and Steel [9]. Again abandon the meanings of "U" and "E" from above. Given that we defined $\mathcal{M}_0^\#$, the reader is sure to be wondering if there is an $\mathcal{M}_1^\#$. Indeed, $\mathcal{M}_1^\#$ is the sharp for the minimal inner model with a Woodin cardinal. The numerology is explained as follows. First, $\mathcal{M}_0$ is L together with the appropriate trivial predicates to turn L into a mouse. Second, $\mathcal{M}_1$ is the canonical $L[E]$ model that plays a role analogous to that of L one level higher in the projective hierarchy in descriptive set theory (see the next paragraph for examples). It turns out that $\mathcal{M}_1$ is the minimal proper class model with a Woodin cardinal and that $\mathcal{M}_1$ is a mouse. Thus $\mathcal{M}_1 = \mathcal{L}[E]$ for a certain sequence E, which is canonical and is to remain fixed for the remainder of this subsection. Let δ_0 be the Woodin cardinal of $\mathcal{M}_1$. It is a fact that every level of $\mathcal{M}_1$ above δ_0 is passive. It is also true that

$$\mathcal{M}_1^\# = \langle L_\nu[E], \in, E \restriction \lambda, U \rangle$$

for some $\nu > \delta_0$ and $U \neq \emptyset$ with $\mathrm{crit}(U) > \delta_0$.

Before continuing, we mention a few connections with descriptive set theory; these results are due to Martin, Steel, and Woodin and can be found in the papers [6] and [9]. In L, the order of construction is a Δ^1_2-definable wellorder of the reals; in $\mathcal{M}_1$, the order of construction is a Δ^1_3-definable wellorder of the reals. The reals of L are exactly those reals that are Δ^1_2 in a code for a countable ordinal; the reals of $\mathcal{M}_1$ are exactly those reals that are Δ^1_3 in a code for a countable ordinal. Actually, it was imprecise to say that

$\mathcal{M}_1$ plays the role of L one level up, since certain phenomena skip a level of the projective hierarchy; in such cases, there is an analogy between L and $\mathcal{M}_2$. For example, $\mathbb{R} \cap L$ is the largest thin Σ^1_2 set of reals, and $\mathbb{R} \cap M_2$ is the largest thin Σ^1_4 set of reals. There are also cases in which the analogy is between L and the so-called *true core model*, K, rather than any of the $\mathcal{M}_n$-s, but we do not go into those here; the reader is directed to Steel [10].

An omission from the previous paragraphs that we wish to be clear about is that it is not possible to capture Woodin cardinals, nor even strong cardinals, by a naïve use of filters as predicates. It was Mitchell who first saw a solution to this problem: the definition of premouse at this level must again be modified in a way that uses extenders instead of filters. In particular, each E_λ should either be the emptyset or it should be an extender over $L_\lambda[E]$.

Towards giving the reader a concrete idea of what an extender is (in a form that was introduced by Dodd and Jensen), recall that if M is a transitive class and $j\colon M \longrightarrow N$ is an elementary embedding with $\operatorname{crit}(j) = \kappa$, then we saw how to factor j through an ultrapower to produce the following commutative diagram.

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr, "i"'] & N \\ & \operatorname{ult}(M, U_j) \arrow[u, "k"'] \end{tikzcd}$$

Here

$$U_j = \{\, A \subseteq \kappa \mid A \in M \text{ and } \kappa \in j(A) \,\}$$

is the M-ultrafilter derived from j and $\operatorname{ult}(M, U_j)$ is the ultrapower of M by U_j. Recall also that $\operatorname{crit}(k) > \kappa$. The greater $\operatorname{crit}(k)$ is, the more $\operatorname{ult}(M, U_j)$ resembles N and i resembles j. Recall that the embedding i fits our needs better than j, even though both are proper classes, because i is definable from the filter U_j, which is a set. On the other hand, j may witness a large cardinal property of κ that is not witnessed by i. We can guarantee more resemblance between i and j by using directed systems of M-ultrafilters derived from j rather than just the single M-ultrafilter U_j. One version of the model theoretic construction is as follows. For each $c \in H^N_{j(\kappa)}$, define

$$F_c = \{\, A \subseteq H^M_\kappa \mid A \in M \text{ and } c \in j(A) \,\}.$$

Then each F_c is an M-ultrafilter. Let $\lambda \leq j(\kappa)$ be a cardinal of N and

$$F = \langle\, F_c \mid c \in H^N_\lambda \,\rangle.$$

Then F is called the *extender of length λ derived from j*. The ultrapowers $\operatorname{ult}(M, F_c)$ for $c \in H^N_\lambda$ together with the natural maps between these ultrapowers form a direct limit system whose direct limit is denoted $\operatorname{ult}(M, F)$. It turns out that there are maps as in the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[dr] & N \\ & \mathrm{ult}(M, F) \arrow[u, "\ell"'] \end{tikzcd}$$

this time with $\mathrm{crit}(\ell) \geq \lambda$.

In summary, one clause in the ultimate definition of premouse is that if $E_\lambda \neq \emptyset$, then $E_\lambda = F$ where F is the extender of length λ derived from an elementary embedding from $L_\lambda[E]$ to some other transitive set. We have left off several other clauses. The kind of iterations that need to be considered also become significantly more complicated: linear iterations are not enough and so-called *iteration trees* must be considered. At this point is seems wise to urge the reader who is interested in learning more to see the articles Löwe-Steel [5] and Steel [8].

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