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REVIEWS

The Association for Symbolic Logic reviews selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Geoffrey Hellman, Thomas Jech, Wolfram Pohlers, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VI 94" refers to the review beginning on page 94 in volume 6 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

JOSEPH R. SHOENFIELD. ***Mathematical logic.*** Republication of JSL XL 234. Association for Symbolic Logic, Urbana, and A K Peters, Natick, Mass., 2001, viii + 344 pp.

This is a photographic copy of the second printing, 1973, of Shoenfield's well-known textbook, first published in 1967. It is surprising that a book this old can still be recommended as a general source at a high level for all major parts of mathematical logic. The first edition was reviewed with a detailed description of the contents in the JOURNAL (JSL XL 234). It may be helpful to summarize briefly that description, with some comments on topics missing because of more recent developments. Basic logic from the proof-theoretic point of view, including Herbrand's theorem, and Henkin's proof of the completeness theorem, is covered in Chapters 1–4. Chapter 5 is titled *The theory of models*, and emphasizes things current in 1967. For stability theory and extensive applications to algebra and analysis one must look elsewhere. Chapters 6–8 are devoted to classical incompleteness theorems and recursive function theory. Chapter 9, titled *Set theory*, is current as of 1967: constructible sets, forcing, and rudimentary large cardinal theory (through Scott's theorem that $V = L$ is inconsistent with the existence of measurable cardinals). The more extensive development of large cardinals, core models, PCF theory, etc., all occurred since 1967. The book is only 344 pages long. It would probably be impossible to have a book of a reasonable size covering the present state of the subject as thoroughly as Shoenfield's did in 1967.

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JON BARWISE and JOHN ETCHEMENDY. ***Language, proof and logic.*** In collaboration with Gerard Allwein, Dave Barker-Plummer, and Albert Liu. CSLI Publications, Stanford, and Seven Bridges Press, New York and London, 1999, xii + 587 pp.

GERARD ALLWEIN, DAVE BARKER-PLUMMER, JON BARWISE, JOHN ETCHEMENDY, and ALBERT LIU. ***LPL software manual.*** CSLI Publications, Stanford, and Seven Bridges Press, New York and London, 1999, vii + 52 pp. + CD-ROM.

Barwise and Etchemendy offer a wonderful taste of what the future of logic instruction could be like. ***Language, proof and logic*** is more than a textbook. It is built around a suite of interactive software which comes bundled with it on CD-ROM: *Tarski's world*, a graphic environment the student uses to learn the basics of the first-order language throughout; *Boole*, used to create and check truth-tables; and *Fitch*, a proof-checking environment in which to construct derivations. But the package also goes beyond local software by offering an active Internet link for both students and instructor to a *Grade grinder* at Stanford. A little more than half the exercises in the book can be completed within the software and submitted to *Grade grinder* by an e-mail link. A uniformly encouraging and impressively detailed response is returned, generally within five minutes or so. The student can choose to have the reply returned to "just me," or to "instructor too," allowing for repeated and revised work before final submission. When I last taught with *Grade grinder* I used a standard spreadsheet to compile final grades from e-mail responses, but it appears that a more complete web-based system for displaying grade reports is now up and running.

What ***Language, proof and logic*** provides is therefore less a traditional textbook than a new technological environment in which to teach introductory logic. At first glance this may seem intimidating, but the package is designed throughout with instructor flexibility in mind. One could easily teach using only the textbook in class, assigning problems that use the software and having students turn in printed or written work. One might also encourage students to send their work to *Grade grinder* as a way of checking and perfecting their work, but insist on a final copy submitted in traditional form. Or one could go the whole route, tracking and grading student work using the Stanford resources as well. Throughout my experience with the package, the people at the other electronic end have been helpful and eager to please, responding immediately to technical glitches and making me feel they were working with me as a logic-teaching team.

My students have enjoyed working within the software. Most readers will be familiar with the graphic environment of *Tarski's world*, in which one can arrange geometric solids in three-dimensional "worlds," writing and checking the truth of symbolic claims about the exhibited world. *Tarski's world* is used here first to introduce a basic language of shapes and locations [$\mathsf{Tet(a)}$, $\mathsf{Small(c)}$, $\mathsf{Between(a,b,c)}$], then the connectives of propositional calculus [$(\mathsf{Cube(a)} \wedge \mathsf{Small(b)}) \rightarrow \mathsf{LeftOf(a,b)}$], and finally the language of quantification [$\forall \mathsf{y}(\mathsf{Dodec(y)} \rightarrow \exists \mathsf{x}(\mathsf{Cube(x)} \wedge \mathsf{LeftOf(x,y)}))$]. It is also used throughout to construct possible worlds as counterexamples in showing non-consequence: that one thing does *not* follow from another. *Boole* is used initially only to check the details of students' hand-constructed truth-tables, but appears later as a tool for instant gratification in constructing more complicated tables. *Fitch* is the best student proof-checker that I have seen anywhere. It is remarkable for the ease with which proofs can be constructed and altered, particularly with an eye to nested conditional proofs. It offers the student quick checks line by line, and indicates by broad category what kind of error is at stake. It avoids much of the mechanical fussiness of proof-checkers I have used to teach logic in the past, which may for example reject a proof without explanation because there is a period missing after a line number. Students seem to like both the friendly interactivity of the software and the fact that it need be no further away, any time of day or night, than their own computer.

Beneath the technological bells and whistles, however, the structure of Barwise and Etchemendy's text is both pedagogically wise and philosophically interesting. For a standard one-semester logic course it will be the first two sections that are used, covering propositional and predicate calculus. (A third section, appropriate for two-semester or graduate courses, covers set theory and metatheoretical results through Löwenheim–Skolem and Gödel incompleteness.) Work is divided throughout into bite-sized sections followed by an immediate handful of exercises; in a standard course two or three of these small sections will be covered each class session. The pace of the book has seemed a good one for my students, always challenging but never so fast as to be incomprehensible. Each exercise seems to teach something new, however small; one never gets the impression that one is facing a page of twenty derivations each of which teaches what one could learn from doing two. The conceptual flow of the book as a whole is smooth and seamless, particularly in the transition from propositional to predicate calculus, and that is a virtue that logic textbooks do not always have.

The book is philosophically interesting as well. As may be evident from the comments above, Barwise and Etchemendy use a simple interpreted language throughout. The student begins with 'Cube(a)' and 'LeftOf(b,c)' as atomic sentences, fully exploiting analytic relations built into the basic predicates: the fact that 'RightOf(c,b)' follows from 'LeftOf(b,c)', for example. Lessons learned here are generalized to other interpreted first-order languages ('Owned(max, scruffy, 2:00)'), and only later to the familiar and fully general P and Q, used in stating all formal rules. Here the philosophical commitment is to meaningful language as the core topic of logic, "in contrast to the so-called 'uninterpreted languages' (surely an oxymoron) so often found in logic textbooks" (p. 15). That philosophical commitment is also intended to have a pedagogical payoff: to make students aware that logic is supposed to be about the meaningful language all around them, rather than some meaningless P's and Q's. Working from the ground up with simple but meaningful examples involving cubes and tetrahedrons avoids much of the bewilderment that faces even some of the best logic students when they are suddenly asked in other texts to do "translations" of unfamiliar epigrams or Biblical passages. It also makes clear the intuitive force behind logical principles that are otherwise in danger of being perceived as purely arbitrary formal rules.

Even the elementary sections of the book include philosophically important material that is unusual in logic textbooks. First-order languages for set theory and arithmetic are introduced in the final sections of the very first chapter, for example, putting '$b \in d$' and '$((1 + 1) \times 1)$' on the same level with 'Cube(a)'. One route offered into the Boolean connectives is a challenge and response game regarding particular arrangements in *Tarski's world* and following patterns of downward saturation familiar from Hintikka and Henkin sets. (So you think Cube(a) $\wedge$ $\neg$Dodec(a) is true, eh? So you think $\neg$Dodec(a) is true? So you think Dodec(a) is false?) A contrast is repeatedly emphasized between logical necessity, consequence, and equivalence in general and the smaller categories of truth-table necessity, consequence, and equivalence. The introduction of conditionals includes a wonderfully appropriate section on Gricean conversational implicature. Technically solid but accessible discussions of soundness and completeness are included at the ends of the sections on propositional and predicate calculus.

Those using the package for the first time should be prepared. The authors are quite clear in the introduction that the book includes more exercises than can be expected in a standard one-semester course. As noted above, each class session will cover two or three sections of the book, and even so one must schedule one's semester carefully in order to include all important material. Instructors should also contact campus technical staff a month or so in advance in order to make sure that computer facilities are appropriately set up. There is no restriction to loading the software on computers campus-wide; student access to *Grade grinder* is always by means of an individualized access code that comes with the book. Our student computing sites at Stony Brook required a hole in the firewall, however, in order to allow proper e-mail interchange with Stanford. It is useful to have either a lab of computers or a projection system available for a few class sessions in order first to introduce students to each of the software resources and to the procedure for "sending," but even this is not absolutely necessary—it is perfectly possible to teach in a standard classroom with student computer work done elsewhere or at home. It should be noted that the system of access codes to *Grade grinder* is designed to keep the whole system profitable by circumventing the used textbook market. Access codes in used textbooks will not be usable.

None of the technological innovations of the package mechanize logic teaching to the point of making it easy. The intent of the *Grade grinder*, the authors note, is to free the instructor from "the most tedious aspect of teaching logic, namely, grading those kinds of problems whose assessment can be mechanized ... formal proofs, translation into FOL, truth tables, and various other kinds of exercises" (p. 12). But the authors clearly recognize that not everything important in logic can be mechanized in this way, and almost half the exercises in the book require careful traditional grading. In the end, I am sure that I do no less work in teaching logic using this package than I have in the past using more traditional texts. What ***Language, proof and logic*** offers is not an easy way to teach logic but an environment of pedagogically well-designed and technologically sophisticated tools, interactive and imaginative, that allow one to teach it better.

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JOHN ETCHEMENDY. ***The concept of logical consequence.*** An unaltered republication of JSL LVII 254. The David Hume series of philosophy and cognitive science reissues. Center for the Study of Language and Information, Stanford 1999, also distributed by Cambridge University Press, New York, vii + 174 pp.

Alfred Tarski (JSL II 83(5)) extended the ideas from his theory of truth (285***16***) to obtain a theory of validity and consequence, defining a *model* of a set of sentences to be a variable assignment that satisfies all the formulas obtained from the sentences by uniformly replacing all the extralogical terms by free variables of appropriate types. A sentence S is *valid* iff every variable assignment is a model of S, and S is a *consequence* of a set of sentences iff every model of the set is a model of S.

John Etchemendy has attacked Tarski's definition ferociously. Much of his criticism focuses on an anomalous feature of Tarski's definition: it does not allow the universe of discourse to vary from one model from another. This anomaly has not been preserved in subsequent uses of the model-theoretic definition of consequence, so it is perhaps a pity that Etchemendy lavishes so much attention on a historical curiosity.

Etchemendy does not discuss a different discrepancy between Tarski's definition and the one employed by contemporary model theory. The former is formulated within the theory of types, the latter in ZFC, taking models to be a kind of structured set. On Tarski's account, the result that every valid sentence of the second-order language of set theory is also true is an immediate application of the logical law of universal instantiation. The truth of every valid sentence is not so obvious if we require our models to be sets. That a sentence of second-order set theory true in every set-sized model is also true of the whole universe is a powerful reflection principle (sometimes called "Kreisel's principle," after JSL XL 499) that goes beyond second-order ZFC.

Tarski saw the central weakness of his theory to be the lack of an account of the distinction between logical and extralogical terms. Etchemendy gives numerous examples to show that an inappropriate choice of logical constants will give outlandish results. He regards these examples as showing Tarski's account to be deeply defective, but perhaps they show only that it needs to be supplemented by an explanation of the logical versus extralogical distinction (something Tarski tried to provide in *What are logical notions?* ***History and philosophy of logic***, vol. 7 (1986), pp. 143–154).

Apart from examples stemming from inflexibility of the universe of discourse and from deviant choices of logical constants, Etchemendy's principal counterexample is this: the continuum problem is a mathematical problem, not a logical problem, yet Tarski treats it as a logical problem, since he can find a second-order formula that is valid iff the continuum hypothesis is true and another that is valid iff it is false. This reviewer is unconvinced, being of the opinion that metalogical questions routinely have mathematical content. For example, the question whether a Diophantine equation has an integer solution is a mathematical question, yet one can reduce it to a metalogical question by finding a formula that is derivable in first-order logic iff an integer solution exists. One cannot, however, reduce the continuum problem to object-level logic by finding a single formula that is valid iff the hypothesis is true and inconsistent iff it is false.

A different characterization of validity treats a sentence as valid iff its truth is ensured by the meanings of the logical terms. It seems clear that a sentence that is valid in this sense is valid in Tarski's sense. If the truth of the sentence is guaranteed by the meanings of the logical terms, then the sentence will remain true if its extralogical terms are assigned new values. The converse is not so clear. If the truth of S is not guaranteed by the meanings of the logical terms, then it is compatible with the meanings of the logical terms that there should be a model in which S is false. How do we conclude from this that such a model actually exists? That is, how do we conclude that S is invalid in Tarski's sense? Perhaps there is an answer, but, as Etchemendy argues forcefully, Tarski has not provided one.

Tarski's definitions of validity and consequence are often regarded as obvious, like "A rhombus is an equilateral parallelogram." If we do not find the proposed counterexamples convincing, we may well regard Tarski's definitions as correct, but after the questions Etchemendy raises, we shall no longer regard them as obvious. If they are true at all, they are deep and dubitable truths, like the Church–Turing thesis. Etchemendy's interpretation of Tarski is unsympathetic, and at times unfair, but it has subjected Tarski's definitions to the philosophical scrutiny they deserve. For this reason, the book has well earned the wide attention it has received since its first edition.

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HAROLD SIMMONS. ***Derivation and computation. Taking the Curry–Howard correspondence seriously.*** Cambridge tracts in theoretical computer science, vol. 51. Cambridge University Press, Cambridge, New York, etc., 2000, xxv + 384 pp.

Although it starts with an introduction to derivation systems and pure λ-calculus, this book is really about applied λ-calculi, which distinguishes it from most introductory texts in the area. Furthermore, Simmons emphasizes learning how to *use* an applied λ-calculus: how typing derivations and reduction interact and, in later chapters, how to use an applied λ-calculus (in this case, a version of Gödel's system T) to define concrete functions. Along the way, of course, metatheoretic issues arise, but they often take a back seat to the business of actually getting things done (e.g. confluence and normalization properties are never proved, though proofs are hinted at in the exercises). This is not inherently bad; after learning from this text, the students should have such a good grasp on the mechanics of the λ-calculus that they expect the basic results to be true rather than observe that they hold. Further, the students can then concentrate on proof techniques rather than the details of the manipulations. Before any more generalities or evaluation, let me give a summary of the contents.

Chapter 1, *Derivation systems*; Chapter 2, *Computation mechanisms*; Chapter 3, *The typed combinator calculus*; Chapter 4, *The typed λ-calculus*. In the first two chapters, Simmons introduces implicational propositional logic via Hilbert- and natural-style systems and the corresponding untyped computation systems of combinator terms and the λ-calculus (here, a *computation* is a formally justified reduction of one term to another). The second two chapters tie these together to form the typed term calculi (only the λ-calculus is used after Chapter 4). The main novelty of these chapters is a time- and space-saving technique for reasoning about derivations and computations that Simmons calls the *arboreal code*. This is a formal term in a language with constructors that correspond to each inference rule. Using the arboreal code, when one defines algorithms, such constructs as "if P is a derivation with last line $\rightarrow$-elimination, major premiss P_1 and minor premiss P_2, then $f(P)$ is ..." are unnecessary; instead, one simply writes "$f(P_1P_2) = \ldots$," where P_1P_2 is the arboreal code for a derivation as just described. This device is important in a text in which almost all results are proved by giving algorithms. For example, subject reduction is not the result that if $\Gamma \vdash t^- : \tau$ and t^- reduces to t^+, then $\Gamma \vdash t^+ : \tau$, but rather an algorithm that maps the given derivation and computation to the new derivation.

Chapter 5, *Substitution algorithms*. Since substitution plays a critical role in the interaction between derivations and computations, it must be handled carefully. With the emphasis on algorithms, Simmons needs a purely mechanical approach to performing substitution. Thus he introduces a syntactic category of *replacements* which implement the approach to substitution described by A. Stoughton (*Substitution revisited*, ***Theoretical computer science***, vol. 59 (1988), pp. 317–325). In addition to being technically smoother to work with when performing substitution over λ-abstractions, replacements have as part of their syntactic data the notion of "safety," which ensures that they work well in the presence of the contexts of typing derivations. Furthermore, by making substitution a syntactic object like derivations and computations, it can now be made part of the data of other algorithms, such as the mutation algorithm of Chapter 6.

Chapter 6, *Applied λ-calculi*. So far, the text has concentrated on the pure λ-calculus. Now Simmons adds atomic types, term constants, and reduction axioms for those constants, to reach the applied λ-calculi. The present chapter is given over to issues that arise for any applied λ-calculus, specifically: type synthesis, mutations, type inhabitation, and computation. Mutations are (yet another) algorithm. Given a derivation of $\Sigma \vdash t : \tau$, a replacement $\mathfrak{a}$, and derivations $\Gamma \vdash s : \sigma$ when $(x : \sigma) \in \Gamma$ and $x\mathfrak{a} = s$, one expects an associated derivation of $\Gamma \vdash t\mathfrak{a} : \tau$. Mutation is the algorithm that does precisely this. Furthermore, by varying the replacement data, one also obtains algorithms showing the admissibility of thinning and exchange. With mutations in hand, β-reduction is handled cleanly. Combined with the reduction axioms, computation is handled as would be expected. I would have liked more discussion of type inhabitation and normalization. It is true that for an arbitrary applied λ-calculus, not much can be said about either topic. I think it would have been helpful, however, to treat at least the latter algorithm for the pure λ-calculus so that readers can get a feel for just how the addition of constants and reduction rules complicate matters significantly.

Chapter 7, *Multi-recursive arithmetic*; Chapter 8, *Ordinals and ordinal notations*; Chapter 9, *Higher order recursion*. The real beauty of this book lies in these last three chapters. Together, they explore the power of Gödel's system T (here called $\boldsymbol{\lambda G}$), along the way developing the basics of hierarchy theory and ordinal notations. In the end, readers have strong intuitions about subrecursion, ordinal notations, and how to use higher types to define complex functions, all in the context of an important, non-trivial applied λ-calculus.

Chapter 7 begins with the definition of $\boldsymbol{\lambda G}$, which is followed by a development of a form of higher-type recursion over *multi-indices*, which are finite sequences of natural numbers. Simmons then turns to the definition of hierarchy generating functions using type-2 and type-3 functionals. Having done so, he turns around and shows how to calibrate the rate of growth of a new function relative to compositions of given functionals. This calibration is naturally associated with a multi-index. Of course, a multi-index is simply another name for an ordinal $< \omega^\omega$ and so the readers have just seen an elegant exposition of how to place a function in the extended Grzegorczyk hierarchy, and now understand why we want to use ordinals to rank even faster-growing functions. Would that I had such an exposition when I first learned subrecursion via higher types! As far as I know, no other text treats this issue so carefully and the book is invaluable for these pages alone. Of course, $\boldsymbol{\lambda G}$ is not forgotten, and Simmons closes with a discussion of implementing all the functionals of the present chapter in the λ-calculus.

Chapter 8 provides a short (mostly) self-contained introduction to ordinals. The interesting part of this chapter is the emphasis on the fact that ordinals are not sufficient for coding up function(al) iteration, because ordinal arithmetic does not behave as we would like. For example, composition corresponds to addition for finite iterates: $f^{m+n} = f^n \circ f^m$. But this fails for transfinite iteration, for we would then have $f^{1+\omega} = f^\omega \circ f$, which is not necessarily f^ω. In the present context, a reasonably simple solution is available: restrict to notations that correspond directly to Cantor normal form, thereby obtaining only one way to write each ordinal. A more general setting would have us define transfinite iteration relative to ordinal *notations*, in which case $f^{1+\omega}$ and f^ω would naturally be different functions, because $1 + \omega$ and ω are different notations. This is a topic which must be left out in the current text, but certainly deserves more attention.

Chapter 9 returns to the task of defining faster and faster-growing functions. Simmons does so in a uniform manner via *long ordinal iterators* $\mathbb{G}\alpha$, *limit creators* (which are functionals that define the limits of ω-sequences), and jump functionals (such as *ack*: $(f, x) \mapsto f^{x+1}x)$. By varying the limit creators and jump functionals, one can define all of the standard transfinite hierarchies. By the technique of lifting a limit creator on a set S to one on $S \to S$ and by proving that the various compositions of the $\mathbb{G}\alpha$ correspond to arithmetic on ordinals, Simmons shows how to define $\mathbb{G}\alpha$ for all $\alpha < \varepsilon_0$. Furthermore, the connection between ordinal exponentiation and increase in type-level is made explicit by the lifting process. Of course, the point throughout is that the various limit creators and jump functionals are trivial to define in $\boldsymbol{\lambda G}$ at any type level. Thus what is left is showing that each $\mathbb{G}\alpha$ can be named, and the inductive definitions show just how to use higher types in order to define faster-growing functions. In the end, we see that $\boldsymbol{\lambda G}$ is powerful enough to name all the generating functions of the $< \varepsilon_0$ levels of the Grzegorczyk hierarchy. At the same time, the readers see that representing the ordinal α in $\boldsymbol{\lambda G}$ really means representing an iteration template defined by α, thereby reinforcing the need to understand ordinal notations at a syntactic level as providing just such templates.

I have not yet mentioned the wealth of exercises in this book. These range from the purely mechanical, making the readers get their hands dirty, through the occasional more sophisticated problems. In keeping with the emphasis on using all the tools presented in the text, the majority of exercises lean toward the former. They may not be "fun," but without all that dirt underneath the fingernails, we have no real understanding of the fruit that we have grown. It is also worth mentioning that almost half of the book is given over to complete solutions to the exercises, which is great for the students learning the material on their own, but not so great for the instructor who wishes to assign problems for graded homework!

Despite the subtitle of the book, it is not clear to me just how we have taken the Curry–Howard correspondence seriously, and what it has gained us. True, we do see it for the pure typed combinator and λ-calculi; but upon first acquaintance, this does not seem to say much more than that we use implicational propositions for our types. I think the most serious omission in the book overall is the lack of a logical formalism that gives rise via a C–H correspondence to the system $\boldsymbol{\lambda G}$. It would certainly take additional time and space; but it is not difficult to do (see, for example, D. Leivant, *Intrinsic theories and computational* *complexity*, ***Logic and computational complexity***, Lecture notes in computer science, vol. 960, Springer, 1995, pp. 177–194) and would leave readers with the understanding that Curry–Howard correspondences have real meat to them.

Returning to generalities, I found this book to be very pleasant to read; the typesetting is good (except for the occasional table that is poorly separated from the preceding text) and there are relatively few typographical errors. Although there is no "declared" audience, anybody wishing to learn the basics of simple type theory will profit from reading it, from advanced undergraduates to researchers. For a graduate course in which one can assume some sophistication among the students, the instructor would probably wish to supplement the material in the book (possibly using another such as Hindley and Seldin's ***Introduction to combinators and λ-calculus***, JSL LIII 985) and cover such topics as confluence, normalization, and maybe also semantics. Of course, a book is a finite object, and many of these topics would have drastically broken the flow of the one under review. But Simmons's book is a wonderful contribution to the literature and must be seriously considered by anybody who wishes to learn or teach the material.

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C. J. Ash and J. Knight. ***Computable structures and the hyperarithmetical hierarchy.*** Studies in logic and the foundations of mathematics, vol. 144. Elsevier, Amsterdam etc. 2000, xv + 346 pp.

Chris Ash started ***Computable structures and the hyperarithmetical hierarchy*** to bring together, in a consistent and compact form, results from computable (recursive, effective) model theory. Much of this work grew out of his collaboration with Julia Knight over the decade of 1985–95. Tragically, Chris Ash's life came to a sudden end in 1995. Julia Knight, working from Ash's outline and pieces of the first five chapters, succeeded in completing this beautiful book.

The book fills an important gap. Although there have been a few Russian monographs, this is the first book from the English-speaking world to cover computable model theory. It generally follows the Western framework of computable mathematics, but includes many Russian results. It focuses on the correspondence between *definability* (syntactic properties) and *computability* (semantic properties) of phenomena in algebraic structures. Naturally, syntactic conditions themselves often involve some form of algorithmic definability which accounts for limitations on complexity on computable isomorphic structures. The Ash–Knight program can be traced back to Ash–Nerode work in 1981.

This is a book about computable and non-computable properties of structures. The structures studied are countable and their languages are computable. Such a structure with a computable domain is *computable* if its atomic diagram is computable. If, moreover, the complete diagram of the structure is computable, then the structure is called *decidable*. While the standard model of arithmetic is computable, its theory, called *true arithmetic*, is not even arithmetical. In addition to complexity of structures and their theories, algorithmic properties studied include complexity of additional relations, and isomorphisms of structures.

The book is self-contained. It has nineteen chapters and an extensive bibliography. The book begins with a survey of fundamental results from computability theory on natural numbers, and from classical model theory. The computational difficulty of structures, their relations, and their isomorphisms is measured within the *hyperarithmetical hierarchy*, introduced by Kleene in the 1950's. In this hierarchy, computational steps are iterated beyond natural numbers by using constructive ordinals, whose notations are recursively defined. *Constructive ordinals* coincide with *computable ordinals*, that is, with the order types of computable well-orderings. Constructive ordinals can also be used to assign computability-syntactic complexity to infinitary first-order formulas. The material on constructive ordinals and computable formulas is presented in detail, in an elegant way, suitable for applications to effective model theory.

In the first part of the book, which includes the first eight chapters, the authors develop tools necessary for the study of computational difficulty of problems arising on algebraic structures. The book starts with a survey of basic computability theory, including the *s-m-n* theorem, the recursion theorem, computably enumerated sets, Turing reducibility, and the arithmetical hierarchy. Then the basic model theory of first-order languages follows, including elementary substructures, consistency criteria, compactness, and prime, saturated, and homogeneous models of theories. The authors further discuss computability-theoretic complexity of theories and structures, and prove Tennenbaum's theorem that there is no computable non-standard model of Peano arithmetic. They then introduce Kleene's ordinal notation and develop the theory of constructive ordinals. They also describe the analytical hierarchy and show that hyperarithmetical sets coincide with Δ^1_1 sets. For a countable language L, the authors define (infinitary) formulas of $L_{\omega_1\omega}$ and their computable analogues. *Computable infinitary formulas* were introduced by Ash in 1986. The authors further present the Makkai consistency criterion, and Scott's isomorphism theorem. They prove a variant of the Barwise compactness theorem and give several of its applications. This part of the book can be used in a course on the hyperarithmetical hierarchy.

The middle part of the book, which consists of Chapters 9–12, begins with a series of results on the existence of computable structures of well-known theories. For example, Khisamiev gave a partial characterization of the reduced Abelian p-groups with computable isomorphic copies, using Ulm invariants. Goncharov and Peretyat′kin independently characterized when a countable homogeneous structure has a decidable isomorphic copy. The book further investigates computability-theoretic complexity of images of a new relation R on the domain of a computable structure $\mathcal{A}$ under isomorphisms from $\mathcal{A}$ to other computable structures. If all such images of R belong to a complexity class $\mathcal{P}$ then R is called *intrinsically* $\mathcal{P}$ *on* $\mathcal{A}$. In the case of non-computable countable structures, we can investigate the corresponding relative versions of algorithmic properties. These relative results are usually simpler and involve the forcing technique, while the results on computable copies often use the priority method. The final chapter in this part (Chapter 12) studies the important notions of computable stability and computable categoricity. A computable structure is *computably stable* if every isomorphism onto another computable structure is computable. Ash and Nerode and independently Goncharov syntactically characterized computable stability. Goncharov also gave a syntactic characterization of computable categoricity. A computable structure $\mathcal{A}$ is *computably categorical* if for every isomorphic computable structure $\mathcal{B}$, there is at least one computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. In this case, we also say that $\mathcal{A}$ and $\mathcal{B}$ have the same computable isomorphism type, or that $\mathcal{A}$ has computable dimension 1. *Computable dimension* of a countable structure is the number of its computable isomorphism types. Computable dimension of familiar mathematical structures, such as linear orderings, Boolean algebras, and Abelian groups, is often 1 or ω. Goncharov constructed, however, a structure of any finite computable dimension.

The last part of the book is the most advanced and complex. It consists of Chapters 13–19. It starts with a gradual exposition of a general, powerful, and intricate machinery for nested priority constructions, invented by Ash in the 1980's and further developed by Ash and Knight and their students. This method is suitable for many computable model-theoretic priority constructions at various levels of the hyperarithmetical hierarchy. The method is formulated in terms of Ash's so-called α-*systems*, where α is a computable ordinal. The authors prove the corresponding abstract metatheorems whose conditions guarantee the success of desired constructions. They further present a number of applications of the metatheorems. These include Barker's result for intrinsically Σ^0_α relations, and Davey's result on inseparability of relations on computable structures. They include Ash's characterizations of stability and categoricity of computable structures in hyperarithmetical degrees, and Ash and Knight's result on pairs of computable structures. These results often require constructions of isomorphisms that exploit the *back-and-forth relations* between finite sequences of elements of structures. The last chapter of the book describes results on complexity of *models of arithmetic*. It presents Harrington's theorem on the existence of a non-standard arithmetical model of Peano arithmetic, whose theory is not arithmetical. Harrington's theorem, for which he developed his worker method, is proved by using an α-system. This chapter also presents Solovay and Marker's characterization of the degrees of non-standard models of true arithmetic as the degrees of enumerations of Scott sets containing the arithmetical sets.

This book is a valuable reference for every logician. The first two parts can also be used in a computable model theory course. In addition to general structures, particular structures whose various algorithmic properties are investigated include equivalence structures, trees, linear orderings and well-orderings, Boolean algebras, models of arithmetic, Abelian groups, fields, and vector spaces. The last part of the book is more advanced and involves general techniques used in current research. The book is well written, with proofs that are intuitively motivated, elegant, and to the point. You will truly enjoy reading it.

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RANDALL DOUGHERTY and ALEXANDER S. KECHRIS. *The complexity of antidifferentiation*. ***Advances in mathematics***, vol. 88 (1991), pp. 145–169.

FERENC BELEZNAY and MATTHEW FOREMAN. *The collection of distal flows is not Borel*. ***American journal of mathematics***, vol. 117 (1995), pp. 203–239.

FERENC BELEZNAY and MATTHEW FOREMAN. *The complexity of the collection of measure-distal transformations*. ***Ergodic theory and dynamical systems***, vol. 16 (1996), pp. 929–962.

HOWARD BECKER. *Pointwise limits of subsequences and $\mathbf{\Sigma}^1_2$ sets*. ***Fundamenta mathematicae***, vol. 128 (1987), pp. 159–170.

HOWARD BECKER, SYLVAIN KAHANE, and ALAIN LOUVEAU. *Some complete $\mathbf{\Sigma}^1_2$ sets in harmonic analysis*. ***Transactions of the American Mathematical Society***, vol. 339 (1993), pp. 323–336.

ROBERT KAUFMAN. *PCA sets and convexity*. ***Fundamenta mathematicae***, vol. 163 (2000), pp. 267–275).

HOWARD BECKER. *Descriptive set theoretic phenomena in analysis and topology*. ***Set theory of the continuum***, edited by H. Judah, W. Just, and H. Woodin, Mathematical Sciences Research Institute publications, vol. 26, Springer-Verlag, New York, Berlin, Heidelberg, etc., 1992, pp. 1–25.

In these seven papers, the authors study the descriptive complexity of various concepts arising in different areas in analysis, such as topology, harmonic analysis, ergodic theory, or the more classical differential calculus. The complexity considered here is relative to the projective hierarchy. We recall that, starting from the class of all Borel subsets of arbitrary Polish spaces, the projective classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ are obtained by alternating the operations of projection and taking complements.

The basic fact of the existence of "true" $\mathbf{\Sigma}^1_1$ sets (i.e., sets that are $\mathbf{\Sigma}^1_1$ and not $\mathbf{\Pi}^1_1$, or equivalently $\mathbf{\Sigma}^1_1$ and not Borel) was the starting point of the theory of analytic sets in 1917. Nevertheless, although most of the fundamental results on the first projective level were achieved before 1925, explicit examples of such sets were so few that one can list them exhaustively. The basic example was given by universal sets, a construction that can be carried over to all projective levels. There was essentially one more example, but specific to the first level, which consisted in some classical version of the modern **WO** (the set of codes of all countable well-orderings). But the first of what one can call "natural examples" is due to Hurewicz in 1930: In the compact space $\mathcal{K}(I)$ of all compact subsets of the unit interval $I = [0, 1]$—equipped with the Hausdorff metric—the set $\mathcal{K}_\omega(I)$ of all countable compact subsets is true $\mathbf{\Pi}^1_1$. The second example, probably more natural for analysts, was given by Mazurkiewicz in 1936: In the Polish space $\mathcal{C} = C(I)$ of all continuous functions on I—equipped with the uniform metric—the set of all everywhere differentiable functions is true $\mathbf{\Pi}^1_1$. For a long time no serious effort was made to enlarge this very short list, mainly from lack of motivation. But more recent developments during the last twenty years elevated complexity results from the status of curiosity to real mathematical statements.

The first problem one encounters when studying the complexity of a concrete set is to give a mathematical meaning for such a study. For this, one has to represent the set under study as a subset of some nice (for example Polish) space, and as we shall see through the coming examples this is not always straightforward. For the rest of this general discussion let A denote a subset of some Polish space, Γ a projective class, and $\check{\Gamma}$ its dual class.

A lower-bound complexity result such as "A is not in $\check{\Gamma}$" can always be viewed as an existential result, since it implies that for any B and C in $\check{\Gamma}$ with $B \subset A \subset C$ the sets $A \setminus B$ and $C \setminus A$ are nonempty. Lower bounds are most often obtained by a completeness argument, which gives in fact a stronger result. We recall that a set A is said to be Γ-complete if every set X in Γ is the inverse image of A by some continuous (or Borel) function. Since there are in Γ sets that are not in $\check{\Gamma}$, a Γ-complete is not in $\check{\Gamma}$, and the converse holds if one admits enough determinacy. In the particular case $\Gamma = \mathbf{\Pi}^1_1$, however, one can also prove sometimes that a set A is not $\mathbf{\Sigma}^1_1$ by a boundedness argument, in connection with the Lusin–Sierpiński index theory, and there are several explicit such situations where it is not known (in ZFC) whether the set A is $\mathbf{\Pi}^1_1$-complete.

An upper-bound complexity result such as "A is in Γ" is a priori easier to prove and, in nice situations, reduced to quantifier-counting, although effective set theory can play a major role in decreasing the expected complexity. A typical example is Kleene's result which asserts that if $P(x, y)$ is $\mathbf{\Pi}^1_1$ then $Q(x)$ defined by "$\exists y \in \Delta^1_1(x), P(x, y)$" is still $\mathbf{\Pi}^1_1$. A more interesting phenomenon, confirmed by practice, is that conversely an amelioration (by any means) of the expected complexity hides some effectiveness property somewhere.

In their paper, Dougherty and Kechris study the complexity of the operation of antidifferentiation. Let Δ denote the set of all derivatives of differentiable functions on I. For any $f \in \Delta$, let $P(f)$ denote the unique primitive F of f satisfying $F(0) = 0$; this defines a mapping $P : \Delta \to \mathcal{C}$. Notice that, by the Lebesgue differentiation theorem, if $f \in \Delta$ is *integrable* then $P(f)(x) = \int_0^x f(t)\,dt$; in fact this integral representation formula was the main motivation for Lebesgue in elaborating his theory of the integral. But in the general case, the problem of computing $P(f)$ for an arbitrary $f \in \Delta$ is much more delicate, and was solved by Denjoy through his theory of totalization, which is essentially a transfinite iterated application of Lebesgue's theorem above, applied on portions of the interval I. As for the complexity of $P : \Delta \to \mathcal{C}$, the first problem is that the set Δ is not endowed with any natural nice topology. But since any $f \in \Delta$ is a first Baire class function on I, there exists some sequence of continuous functions $\bar{g} \in \mathcal{C}^\omega$ that converges pointwise to f; we shall then say that $\bar{g}$ is a code for f and write $f = L(\bar{g})$. Hence if $\bar{\Delta}$ is the set of all codes for functions in Δ then the pointwise limit operator $L : \bar{\Delta} \to \Delta$ defines a parameterization of Δ by a subset of the Polish space $\mathcal{C}^\omega$, and one can consider the mapping $\bar{P} = P \circ L : \bar{\Delta} \to \mathcal{C}$. By a result of Ajtai, $\bar{\Delta}$ is a $\mathbf{\Pi}^1_1$ set and $\bar{P}$ is a $\mathbf{\Delta}^1_1$ partial function on $\mathcal{C}^\omega$; this is proved by analyzing Denjoy's process of totalization. The authors prove that $\bar{P}$ is not $\mathbf{\Delta}^1_1$, in other words that $\bar{P}$ is not the restriction to $\bar{\Delta}$ of a $\mathbf{\Delta}^1_1$ mapping defined on $\mathcal{C}^\omega$. Notice that the distinction here between $\mathbf{\Delta}^1_1$ and Borel is due to the fact that the domain $\bar{\Delta}$ is not $\mathbf{\Sigma}^1_1$. An informal translation of this mathematical statement is that recovering the primitive of $f \in \Delta$ in the general case, even when a code $\bar{f} \in \bar{\Delta}$ of f is given, cannot be obtained by an analytical formula from $\bar{f}$ and hence from f. This also means that the computation of the primitive (not only totalization) from any Borel data has to refer in one way or another to arbitrary countable ordinals. The proofs make substantial use of recursiveness on the integers, and effective set theory in recursively presented spaces. In fact the main theorem of the paper is a representation result of any hyperarithmetical real number as the evaluation $F(1)$ of some differentiable function F on I, whose derivative $f \in \Delta$ admits a recursive code $\bar{f} \in \bar{\Delta} \subset C^{\omega}$; moreover by Matiyasevich's results on recursively enumerable sets of integers, f can be taken in the form $f = \sum_{n=0}^{\infty} f_n$ with explicit elementary functions f_n.

The next two papers from the above list, by Beleznay and Foreman, deal with more elaborate concepts. The first of these two papers has its roots in topological dynamics. A *flow* is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a homeomorphism on X. The flow is said to be *distal* if T satisfies the following strong form of injectivity: For any two distinct points $x \neq y$ in X, the infimum of all distances $d(T^n x, T^n y)$ for $n \in \mathbb{Z}$, is > 0. A very simple example of a distal flow is given by an isometry, that is, a mapping $T : X \to X$ which preserves the metric ($d(Tx, Ty) = d(x, y)$ for all x and y in X). Furstenberg defined the notion of *isometric extension* of a distal flow (X, T); this is a distal flow (Y, S) for which the initial flow (X, T) appears as a kind of quotient structure. Isometries are precisely the isometric extensions of the *trivial flow* (the unique flow on a one-point space $X = \{a\}$). The crucial result in this topic is a theorem of Furstenberg from which it follows that any distal flow can be obtained, in a canonical way, from the trivial flow by transfinite countably many iterations of isometric extensions. In particular to any distal flow one can associate some countable ordinal, called its Furstenberg order. Answering an open question, the authors prove that any countable ordinal can arise in this procedure. They also prove that for an adequate representation of flows as elements in some nice space, the set $\mathcal{D}$ of all distal flows is $\mathbf{\Pi}^1_1$-complete (hence not Borel), and that the Furstenberg order is a $\mathbf{\Pi}^1_1$ *norm* on $\mathcal{D}$. Their second paper concerns the notion of *measure-distal transformations*, a measure-theoretic analogue to the notion of distal flow. The results are very similar to the topological case, although the technical material is very different and involves a non-negligible amount of functional analysis. In both of these two works, descriptive set theory is used mainly as a language, although some deep theorems are used, and the heart of the proofs is most often on the analysis side. Each of the two papers, however, starts with a general self-contained presentation of the topic, which includes main known results and techniques needed, and a motivated non-specialist reader can follow all details.

The joint paper by Becker, Kahane, and Louveau (denoted by [BKL]) is a continuation of Becker's first paper (denoted by [B]). Both papers concern the study of projective sets of level 2 in functional spaces. The main result of [B] is a representation theorem: Any $\mathbf{\Sigma}^1_2$ subset R of $\mathcal{C}$ can be represented in the form $R = \{g \in \mathcal{C} : g$ is the pointwise limit of some some subsequence of $\bar{f}\}$ for some fixed sequence $\bar{f} \in \mathcal{C}^{\omega}$. It follows then (from a slightly more precise version of this result) that the set $S = \{\bar{f} \in \mathcal{C}^{\omega}$: some subsequence of $\bar{f}$ pointwise converges to some $g \in \mathcal{C}\}$ is $\mathbf{\Sigma}^1_2$-complete, and that the set $S' = \{\bar{f} \in \mathcal{C}^{\omega}$: for any $g \in \mathcal{C}$ there is some subsequence of $\bar{f}$ that converges pointwise to $g\}$ is $\mathbf{\Pi}^1_3$-complete. In [BKL] various modifications of the set S are considered and proved to be also $\mathbf{\Sigma}^1_2$-complete. The proofs are based on another representation result: Any $\mathbf{\Sigma}^1_2$ subset A of $\mathbf{2}^{\omega}$ can be represented in the form $A = \{\alpha \in \mathbf{2}^{\omega}$: some subsequence of $\langle F_n(\alpha, \cdot)\rangle_n$ converges pointwise to $0\}$ for some sequence $\langle F_n\rangle$ of continuous functions on $\mathbf{2}^{\omega} \times \mathbf{2}^{\omega}$, where $F_n(\alpha, \cdot)$ denotes the "partial" function defined on $\mathbf{2}^{\omega}$ (actually each F_n is the characteristic function of some clopen set). From this, they derive that a number of families of compact sets, considered in harmonic analysis, are $\mathbf{\Sigma}^1_2$-complete; among these are the families of the so-called D-sets and N_0-sets, which are compact subsets of I defined by some behavior of the sequence (or the series)

$\langle \sin 2\pi nx \rangle$. In this direction, one should also mention Kaufman's paper, which studies some $\mathbf{\Sigma}^1_2$-complete subsets of the classical Banach space c_0, related to convexity.

Finally we come to Becker's second paper which is in fact a survey on complexity results. The author lists a large number of examples of natural sets for which the exact complexity is known. Proofs are omitted but the paper contains many interesting comments on various complexity problems. To give a flavor to the reader, a small selection of examples is presented here. For simplicity we define the sets only through analysis, the topology being most often suggested by the context. Examples of complete $\mathbf{\Pi}^1_1$ sets include (1) the set of all functions $f \in C(I)$ that are nowhere differentiable (Mauldin); (2) the set of all functions $f \in C(I)$ with Fourier series converging everywhere (Ajtai–Kechris); (3) the set of all $f \in L^1(I)$ with Fourier series diverging everywhere (Kechris); (4) the set of all $K \in \mathcal{K}(\mathbb{R}^2)$ that are simply connected (Becker). As opposed to the last example, for $n \geqq 4$, the set of all $K \in \mathcal{K}(\mathbb{R}^n)$ that are simply connected is $\mathbf{\Pi}^1_2$-complete (Becker); but this is not known for $n = 3$. The paper also contains several similar open questions. For example the author mentions that no natural example of (exactly) the fourth level is known. The following is also another interesting problem presented in the paper: Consider again the $\mathbf{\Pi}^1_3$ set $S' \subset \mathcal{C}^\omega$ defined in the previous paragraph and let $S'' = \{\bar{f} \in \mathcal{C}^\omega$: there is exactly one $g \in \mathcal{C}$ such that no subsequence of $\bar{f}$ converges pointwise to $g\}$. Assuming $\mathbf{\Delta}^1_2$-determinacy, one can prove that (a) S'' is $\mathbf{\Pi}^1_3$; (b) there is no $\mathbf{\Delta}^1_3$ set separating S' from S''. But while the conjunction (a) *and* (b) is false if $V = L$, it is not known whether either (a) or (b) is provable in ZFC. The author explicitly states in the introduction of his work that the survey is not exhaustive. Still, one can regret that many recent complexity results in Banach space theory are not presented in this survey. Aside from studying, as in Kaufman's paper, the complexity of subsets of some fixed Banach space, one can also study the complexity of families of Banach spaces (see Benoît Bossard, *Codage des espaces de Banach séparables, Familles analytiques ou coanalytiques d'espaces de Banach*, ***Comptes rendus de l'Académie des Sciences***, Série I, Mathématique, vol. 316 (1993), pp. 1005–1010). The reader can also find more works in this direction among the references in Kaufman's paper.

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TOMEK BARTOSZYNSKI. *On the structure of measurable filters on a countable set*. ***Real analysis exchange***, vol. 17 no. 2 (1992), pp. 681–701.

TOMEK BARTOSZYNSKI and SAHARON SHELAH. *Intersection of $< 2^{\aleph_0}$ ultrafilters may have measure zero*. ***Archive for mathematical logic***, vol. 31 (1992), pp. 221–226.

TOMEK BARTOSZYNSKI and HAIM JUDAH. *Measure and Category—filters on ω*. ***Set theory of the continuum,*** edited by H. Judah, W. Just, and H. Woodin, Mathematical Sciences Research Institute publications, vol. 26, Springer-Verlag, New York, Berlin, Heidelberg, etc., 1992, pp. 175–201.

TOMEK BARTOSZYNSKI, MARTIN GOLDSTERN, HAIM JUDAH, and SAHARON SHELAH. *All meager filters may be null*. ***Proceedings of the American Mathematical Society***, vol. 117 (1993), pp. 515–521.

TOMEK BARTOSZYŃSKI. *Remarks on the intersection of filters*. ***Topology and its applications***, vol. 84 (1998), pp. 139–143.

This series of papers finds its roots in the fascinating interplay between measure and category, a treasure-land of subtle combinatorial arguments and challenging problems. Filters on the natural numbers form another area immensely rich in combinatorics, and this particular sequence of papers brings the two together with some deep insight and ingenuity.

Because of their specific nature as sets of reals, filters exhibit particular measure and category behavior. It was already known for example by Sierpiński that a filter has either measure zero or is non-measurable, and that it is either meager or does not have the Baire property. The combinatorial structure of meager filters was later successfully described by Michel Talagrand (*Compacts de fonctions mesurables et filtres non mesurables*, ***Studia mathematica***, vol. 67 (1980), pp. 13–43) and that of the more subtle measurable filters by Bartoszynski in the first paper above. Talagrand had also shown that the intersection of countably many non-measurable filters is a non-measurable filter, and similarly for category, that the intersection of countably many filters without the Baire property is a filter without the Baire property. The anticipation was obvious after he had shown that under Martin's axiom, the intersection of fewer than $2^{\aleph_0}$ filters without the Baire property is a filter without the Baire property. But it is Fremlin who triggered the typical and fascinating measure and category rivalry by showing that Martin's axiom provides a family of $2^{\aleph_0}$ non-measurable filters such that every uncountable subfamily has a measurable intersection (see the third paper). I am sure this was enough bait to spark the interest of many.

Bartoszynski took the lead by collecting his immeasurable expertise of measure. He first brought back from *On covering of real line by null sets* (***Pacific journal of mathematics***, vol. 131 (1988), pp. 1–12) his generalization of the usual measure on 2^ω to the measures $\mu_{\hat{p}}$, and produced a combinatorial characterization of $\mu_{\hat{p}}$-measurable filters. These new measures depend on a sequence of reals $\hat{p} = \{p_n : n \in \omega\}$ such that $p_n \in \left(0, \frac{1}{2}\right]$, and the usual measure is the one given the sequence $p_n = \frac{1}{2}$ for all n. His first breakthrough was, under Martin's axiom, that the intersection of fewer than $2^{\aleph_0}$ $\mu_{\hat{p}}$-non-measurable filters is $\mu_{\hat{p}}$-non-measurable, assuming that $\lim_{n\to\infty} p_n = 0$. This is essentially the content of the first paper.

A related direction required the filters to be maximal, the so-called ultrafilters. Szymon Plewik (*Intersections and unions of ultrafilters without the Baire property*, ***Bulletin of the Polish Academy of Sciences***, Mathematics, vol. 35 (1987), pp. 805–808) had shown without extra assumptions that the intersection of fewer than $2^{\aleph_0}$ ultrafilters did not have the Baire property, and Fremlin naturally asked the related problem about measure. In the second paper above, this was shown not to be the case, where a model of ZFC is obtained containing a family of fewer than $2^{\aleph_0}$ ultrafilters having measurable intersection.

The third paper contains a good survey of the area, but essentially no new results. It follows from the results of the remaining papers that the open problem whether every filter having the Baire property extends to a non-measurable filter having the Baire property is independent (of the set theory axioms ZFC).

Talagrand had also shown (op. cit.) that there is a measurable filter without the Baire property, and Bartoszynski actually proved a stronger result that every measurable filter can be extended to a measurable filter without the Baire property. This again begs for the dual problem whether a non-measurable filter with the Baire property always exists. This is soon answered in the negative by a proof in the fourth paper of the relative consistency that all meager filters have measure zero. This is done by a fine analysis of the Cohen model, a key player together with the random-real model of the measure versus category saga.

In the last paper, the author undertakes a more detailed analysis using known cardinal invariants of the continuum. In particular, it is shown that the intersection of fewer than $\mathfrak{t}$ filters without the Baire property is a filter without the Baire property. It is also shown that some family of $\aleph_1$ non-meager filters having a meager intersection can consistently exist in conjunction with a strong version of Martin's axiom (for Suslin partial orders). This leaves open the characterization of the smallest size of such a family as a cardinal invariant.

The intriguing measure and category adventure continues.

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JEREMY AVIGAD. *Formalizing forcing arguments in subsystems of second-order arithmetic*. ***Annals of pure and applied logic***, vol. 82 (1996), pp. 165–191.

In this interesting paper, known conservation results are obtained through a new technique that provides additional proof-theoretic information. The subsystems of second-order arithmetic involved are WKL_0 and $\mathsf{WKL}+_0$, which allow the development of a fair amount of ordinary mathematics. The latter fact has been substantiated by research in *reverse mathematics* (see Simpson's monograph ***Subsystems of second order arithmetic***, JSL LXIV 1356). Simpson has argued in *Partial realizations of Hilbert's program* (***The journal of symbolic logic***, vol. 53 (1988), pp. 349–363) that, since the conservation results show that these systems are proof-theoretically weak, this leads to a partial realization of Hilbert's program.

WKL_0 extends the base theory RCA_0 (which consists of the arithmetic axioms, $\mathbf{\Sigma}^0_1$ induction, and $\mathbf{\Delta}^0_1$ comprehension) by König's lemma restricted to trees of sequences of 0's and 1's. In *The Baire category theorem in weak subsystems of second-order arithmetic* (***The journal of symbolic logic***, vol. 58 (1993), pp. 557–578), Brown and Simpson introduced the scheme BCT asserting that every arithmetically defined sequence of dense open sets of Cantor space has non-empty intersection, and used it to prove a form of the Baire category theorem and some of its consequences. $\mathsf{WKL}+_0$ consists of WKL_0 plus BCT.

H. Friedman (who introduced RCA_0 and WKL_0) proved by model-theoretic techniques that WKL_0 is conservative over PRA (primitive recursive arithmetic) for $\mathbf{\Pi}^0_2$ sentences, and later Harrington (see Simpson's book) used a forcing argument to show that WKL_0 is conservative over RCA_0 for $\mathbf{\Pi}^1_1$ formulas; Brown and Simpson extended Harrington's result to $\mathsf{WKL}+_0$, again using forcing. From a proof-theoretic perspective, Sieg, in *Fragments of arithmetic* (JSL LII 1054) gave an effective version of Friedman's result using cut-elimination, while Hájek, in *Interpretability and fragments of arithmetic* (JSL LX 1014) showed how to interpret WKL_0 in the fragment of first-order arithmetic $I\Sigma_1$. A consequence of the latter fact is that there is no significant increase in the length of proofs from WKL_0 to RCA_0. The latter fact does not follow from Sieg's results, and indeed there is a superexponential increase in the length of proofs between RCA_0 and PRA, as shown in Ignjatović's Ph.D. dissertation (*Fragments of first and second order arithmetic and length of proofs*, University of California, Berkeley, 1990).

Avigad proves Hájek's no-increase result through a different technique and extends it to $\mathsf{WKL}+_0$, obtaining for the first time an effective proof of Brown and Simpson's conservation result. The precise statement of the main result is that there exist a recursive function f and a polynomial p such that if d codes a proof in $\mathsf{WKL}+_0$ of a $\mathbf{\Pi}^1_1$ formula φ, then $f(d)$ codes a proof in RCA_0 of φ and $|f(d)| < p(|d|)$, where $|d|$ is the number of symbols occurring in the proof coded by d.

The theorem is very interesting, but even more important is its proof: Avigad shows how to express syntactically forcing notions within a weak base theory (namely RCA_0). Since iterated forcing is used in both Harrington's and Brown and Simpson's proofs, this requires a quite careful analysis of the complexity of the formulas involved. Avigad's approach is quite general and can probably be applied in other cases as well. The formalization of forcing within subsystems of second-order arithmetic had previously been pursued by the reviewer in *Borel quasi-orderings in subsystem of second-order arithmetic* (***Annals of pure and applied logic***, vol. 54 (1991), pp. 265–291), but in that paper goals and techniques were quite different (viz. the formalization of proofs in descriptive set theory that employ Gandy forcing and the use of inner models) and the systems involved were much stronger (ACA_0 and above).

The paper is well organized and nicely written: Avigad clearly states the idea of the proof, highlights the difficulties arising in its realization, and explains carefully the way he overcomes them; he describes in detail the general framework, applies it to the simplest cases (starting with non-iterated forcing), and eventually reaches the more complex applications, up to the result about $\mathsf{WKL}+_0$. There are a few annoying misprints, the most eye-catching being the lack of limitation on the complexity of the formula occurring in the scheme BCT (φ must be arithmetical), and the statement of Lemma 9.3, where two conditions are identical (one of them should be $p \Vdash \sigma \subset \hat{G}$).

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MITSURU TADA and MAKOTO TATSUTA. *The function $\lfloor a/m \rfloor$ in sharply bounded arithmetic*. ***Archive for mathematical logic***, vol. 37 no. 1 (1997), pp. 51–57.

In his ***Bounded arithmetic*** (JSL LVI 759), Samuel Buss introduced the hierarchy of bounded theories of arithmetic S_2^i $(i = 0, 1, 2, \dots)$. For $i > 0$, these theories are conspicuously related to important classes of computational complexity. For instance, the provably total functions (with appropriate graphs) of S_2^1 are exactly the polynomial-time computable functions.

The paper under review concerns the system S_2^0. Based on the obvious observation that the predecessor of a number whose binary representation has *few* ones may have *many* ones in its binary representation, Gaisi Takeuti was able to prove (*Sharply bounded arithmetic and the function $a \dot{-} 1$*, JSL LXI 697) that the (modified) predecessor function $a \dot{-} 1$ is not provably total in S_2^0. Of course, the above observation lacks a clear meaning. Takeuti's rigorous proof relies on a clever interpretation of S_2^0 into $S_2 = \bigcup_i S_2^i$ together with a suitable use of Parikh's bounding theorem. For m a (standard) natural number that is not a power of 2, Tada and Tatsuta show that there are natural numbers a whose binary representation have *few* ones but for which the binary representations of $\lfloor a/m \rfloor$ have *many* ones. Using Takeuti's above-mentioned method, they are able to show that the function $\lfloor a/m \rfloor$ is not provably total in S_2^0. On the other hand, for m a power of 2, they easily show that the function $\lfloor a/m \rfloor$ is provably total in S_2^0.

Given that S_2^0 does not correspond to (nor meshes well with) known computational complexity classes, why should one study S_2^0 and its provably total functions? The authors state that "S_2^0 will give us a better understanding of the systems S_2^i and will possibly give us also knowledge about the theory of computational complexity." That notwithstanding, the reviewer believes that S_2^0 is simply a by-product of a convenient weak base theory over which one can set up the robust and important systems S_2^i $(i > 0)$. Suitable variations of S_2^0 are forthcoming for other convenient weak base theories. Seemingly, the provably total functions of these variants are very sensitive to the variations themselves. The present reviewer is not aware of a convincing argument for the pertinence of S_2^0 as an independent theory.

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WILFRIED BUCHHOLZ. *An intuitionistic fixed point theory*. ***Archive for mathematical logic***, vol. 37 no. 1 (1997), pp. 21–27.

Fixed-point theories are extensions of classical or intuitionistic arithmetic, by axiom schemes asserting that fixed points exist for definable, positive set operators on natural numbers. The assumption that these fixed points are minimal leads to the theories of inductive definitions, well studied within proof theory (Buchholz, Feferman, Pohlers, and Sieg, ***Iterated inductive definitions and subsystems of analysis***, JSL LIX 668). The intuitionistic and classical versions of these theories have the same proof-theoretic strength and can express transfinite induction over higher number classes $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \dots$ as formulated by Church and Kleene. Surprisingly, if we assume only fixed points, this harmony breaks down as is shown in the paper under review. The classical version of the n-times iterated fixed-point theory $\widehat{\mathrm{ID}}_n$ was shown by Aczel $(n = 1)$ and Feferman $(1 < n < \omega)$ to have the same strength

α_n as Martin-Löf type theory with n universes ($\alpha_0 = \varepsilon_0$, $\alpha_{n+1} = \varphi_{\alpha_n}0$, where φ is the two-place Veblen function). Buchholz shows (Theorem 2) that the intuitionistic version of $\widehat{\mathrm{ID}}_1$ is conservative over intuitionistic arithmetic (HA) for almost negative formulas, and thus is weaker than its classical version. The idea is to interpret the theory into HA extended with Church Thesis (CT_0), which asserts that there is a recursive choice function for every $\forall\exists$ combination. The fixed point of a strongly positive operator form $\Phi(P, x)$ is constructed by setting up a certain recursive infinitary proof system. A crucial rule is that from $\Phi(P, \underline{n})$ we may infer $P(\underline{n})$, so its proofs may be non-well-founded. Indeed a recursive construction, enabled by CT_0, is used to prove that the set of n such that $P(\underline{n})$ has a (non-well-founded) proof, is a fixed point of the operator. The theory $\mathrm{HA} + \mathrm{CT}_0$ is then reduced to HA via a realizability interpretation, which explains the restriction to almost negative formulas in the conservation result. In a subsequent paper T. Arai (BSL VII 77) extends the conservation result to finitely iterated fixed-point theories, and can avoid the formula restriction by using other means.

In view of these results, one may wonder whether first-order axiomatizations of fixed-point theories are appropriate for the constructive case. On the other hand, the universes of Martin-Löf type theory can be regarded as fixed points to certain operators on families of types, and in this situation they lead to considerable gain in proof-theoretic strength. Type theories ranging from predicative to impredicative, in strength well beyond second-order arithmetic, may be formulated by using such fixed points.

A positive application of the conservation result is provided in the article. The intuitionistic version of $\widehat{\mathrm{ID}}_1$ is suggested as a natural metatheory for infinitary proofs. A relativized version of Theorem 2 and an internalized Gödel–Gentzen interpretation (Theorem 3) is then used to show how Schütte-style cut-elimination proofs for a theory Th formally yield lower bounds on the proof-theoretic strength of Th.

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