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POLISH METRIC SPACES: THEIR CLASSIFICATION AND ISOMETRY GROUPS

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§1. Introduction. In this communication we present some recent results on the classification of Polish metric spaces up to isometry and on the isometry groups of Polish metric spaces. A *Polish metric space* is a complete separable metric space (X, d).

Our first goal is to determine the exact complexity of the classification problem of general Polish metric spaces up to isometry. This work was motivated by a paper of Vershik [1998], where he remarks (in the beginning of Section 2): "The classification of Polish spaces up to isometry is an enormous task. More precisely, this classification is not 'smooth' in the modern terminology." Our Theorem 2.1 below quantifies precisely the enormity of this task.

After doing this, we turn to special classes of Polish metric spaces and investigate the classification problems associated with them. Note that these classification problems are in principle no more complicated than the general one above. However, the determination of their exact complexity is not necessarily easier.

The investigation of the classification problems naturally leads to some interesting results on the groups of isometries of Polish metric spaces. We shall also present these results below.

The rest of this section is devoted to an introduction of some basic ideas of a theory of complexity for classification problems, which will help to put our results in perspective. Detailed expositions of this general theory can be found, e.g., in Hjorth [2000], Kechris [1999], [2001].

In mathematics one frequently deals with problems of classification of various objects up to some notion of equivalence by invariants. Quite often these objects can be viewed as forming a definable (Borel, analytic, etc.) subset X of a *standard Borel space* $\hat{X}$ (i.e., a Polish space with its associated σ-algebra of Borel sets), and the equivalence relation as a definable (Borel, analytic, etc.) equivalence relation E on X. A complete classification of X

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up to E consists then of finding a set of invariants I and a map $c : X \to I$ such that, for all $x, y \in X$,

$$xEy \Leftrightarrow c(x) = c(y).$$

For this to be of interest both I and c must be as simple and concrete as possible.

For our purposes, the simplest case is when the invariants are concrete enough so that they can be represented as elements of a standard Borel space (and the map c is fairly explicitly definable). More precisely let us call E (and the classification problem it represents) *concretely classifiable* (or *smooth* or *tame*) if there is a standard Borel space Y and a Borel (measurable) map $c : X \to Y$ such that $xEy \Leftrightarrow c(x) = c(y)$.

To apply these ideas to the classification problem of Polish metric spaces up to isometry, we first indicate how we view any such space as an element of a standard Borel space, in other words we describe a standard Borel space of Polish metric spaces. One natural way to do that is the following: Fix a universal Polish metric space, like the *Urysohn space* $\mathbb{U}$ (which we will discuss more in §2 below). Then every Polish metric space is, up to isometry, a closed subspace of $\mathbb{U}$, and we can view $F(\mathbb{U})$, the standard Borel space of closed subsets of $\mathbb{U}$ with the Effros Borel structure (see Kechris [1995], 12.C), as the *space of Polish metric spaces*. Denote then by $\cong_i$ the equivalence relation of isometry between metric spaces. Our objective is to understand the complexity of $\cong_i$ on Polish metric spaces, or, equivalently, closed subsets of $\mathbb{U}$.

First let us note that if we restrict $\cong_i$ to the space $K(\mathbb{U})$ of compact subsets of $\mathbb{U}$, in other words if we consider the isometry problem for compact metric spaces, then already Gromov (see, e.g., Gromov [1999], $3.11\frac{1}{2}_+$ or 3.27) has shown that it is concretely classifiable. However, as Vershik [1998] points out, the classification of general Polish metric spaces up to isometry is not concretely classifiable, thus quite complicated, in some sense. But can we make this more precise and calculate how complicated it really is? This is the problem that we address in the next section.

To arrive at an answer, one first has to define in what sense a classification problem is at most as complicated as another. This is made precise by means of the concept of reducibility between equivalence relations. If E, F are equivalence relations on subsets X, Y, resp., of standard Borel spaces, we say that E is *Borel reducible* to F, in symbols,

$$E \leq_B F$$

if there is a Borel map $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

Intuitively, this means that any complete invariants for F work as well for E (after composing with f) and therefore, in some sense, the classification problem represented by E is at most as complicated as that of F. Also E is *Borel bireducible* with F, in symbols

$$E \sim_B F \Leftrightarrow E \leq_B F \ \& \ F \leq_B E,$$

means that the classification problems represented by E, F have the same complexity. Finally,

$$E <_B F \Leftrightarrow E \leq_B F \ \& \ F \not\leq_B E,$$

signifies that the classification problem of E is strictly simpler than that of F.

The (partial pre-)order $\leq_B$ imposes a hierarchy of complexity on classification problems and our goal here is to find the place of the isometry problem of Polish metric spaces in this hierarchy. In the study of this subject several important benchmarks have been discovered, which can be used to calibrate the difficulty of specific classification problems that come up in various fields of mathematics. We will review the ones that are relevant to us here. See Becker and Kechris [1996] for more details.

For any Polish group G and Borel action $(g, x) \mapsto g \cdot x$ of G on a standard Borel space X (a *Borel G-space* for short) we denote by E_G^X the corresponding orbit equivalence relation

$$x E_G^X y \Leftrightarrow \exists g (g \cdot x = y).$$

(This is an analytic but not always Borel equivalence relation.) It turns out that among all E_G^X, with G fixed, there is a most complex, i.e., universal, one. In other words, there is a Borel G-space X such that for all Borel G-spaces Y we have $E_G^Y \leq_B E_G^X$. It is unique up to $\sim_B$ and we denote it by E_G^∞.

Now, letting G vary over all Polish groups, there is a universal relation of the form E_G^∞. This is again unique up to $\sim_B$ and we call it the *universal equivalence relation induced by a Borel action of a Polish group*. It can be realized by E_G^∞, where G is either the homeomorphism group of the Hilbert cube or the isometry group of the Urysohn space. This follows from the results of Uspenskiĭ [1986],[1990] that these groups are universal Polish groups, i.e., contain every Polish group as a closed subgroup. In many ways, that the theory of Borel reducibility makes precise, the univeral equivalence relation induced by a Borel action of a Polish group is an enormously complex equivalence relation.

Let S_∞ be the infinite symmetric group of all permutations of $\mathbb{N}$, with the topology of pointwise convergence. We can also consider $E_{S_\infty}^\infty$, the universal equivalence relation induced by a Borel action of S_∞. This is much smaller, in terms of the ordering $<_B$, than the universal equivalence relation induced by a Borel action of a Polish group. Nevertheless, it is already not concretely classifiable. An important concrete realization of $E_{S_\infty}^\infty$ is *graph isomorphism*, the isomorphism relation between countable graphs (see Becker and Kechris [1996]).

Among the equivalence relations of the form E_G^∞, where G varies over all *countable* groups (they are Polish with the discrete topology), there is also a universal relation, which we denote by E_∞. Again E_∞ is much smaller, in terms of $<_B$, than graph isomorphism. Yet it is still not concretely classifiable. Recall that a Borel equivalence relation is *countable* if all of its equivalence classes are countable. By a theorem of Feldman and Moore (see, e.g., Dougherty, Jackson, and Kechris [1994]), every countable Borel equivalence relation can be viewed as the orbit equivalence relation of a Borel action of a countable group. Thus E_∞ is also universal among countable Borel equivalence relations. A realization of E_∞ is $E_{F_\infty}^\infty$, where F_∞ is the free group with $\aleph_0$ many generators (see Dougherty, Jackson, and Kechris [1994]). It is a result of Jackson-Kechris-Louveau (see, e.g., Hjorth and Kechris [1996], p.241), that E_∞ is Borel bireducible with the isomorphism relation of countable connected locally finite graphs (i.e., graphs in which every vertex has finitely many neighbors) and also with the isomorphism relation of countable locally finite trees (i.e., connected acyclic graphs).

We shall show in the sequel that these benchmark equivalence relations, which play important roles in the general theory of equivalence relations, come up naturally in our study of the classification problem of Polish metric spaces and serve as quantitative measures for the complexity of the equivalence relations involved.

The results presented in this announcement come from two separate sources, the work of the first author (Clemens [2001]), and the joint work of the second and the third authors (Gao and Kechris [2000]). For brevity we shall cite Clemens [2001] as (Cl) and Gao and Kechris [2000] as (GK) in the subsequent sections. These two papers, whose work was done independently, have only one overlapping result (which we will discuss in §2 below) and use quite different methods, so essentially complement each other. Taken together they provide a more complete picture of the subject we study here.

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§2. The classification of general Polish metric spaces. Our first result computes the complexity of the isometric classification of Polish metric spaces as being exactly that of the universal equivalence relation induced by a Borel action of a Polish group. More precisely:

THEOREM 2.1. *The equivalence relation of isometry of Polish metric spaces, $\cong_i$, is Borel bireducible with the universal equivalence relation induced by a Borel action of a Polish group.*

In the rest of this section we will sketch the proof of Theorem 2.1. A complete proof can be found in (GK). One direction of the theorem, that

$E_G^X \leq_B \cong_i$ for every Borel G-space X, is independently proved in (Cl) by a different method. We shall sketch both methods below. Further results presented in subsequent sections are mostly proved by variations of one of the methods we sketch here.

The Urysohn space $\mathbb{U}$ plays a crucial role in the arguments of (GK). This space was introduced by Urysohn [1927] and further studied in Katětov [1988], Uspenskiĭ [1990], Vershik [1998], and Gromov [1999] $3.11\frac{2}{3}{}_{+}$.

A separable metric space M is called *Urysohn* if for any finite metric space X and any subspace $Y \subseteq X$ every isometric embedding $f : Y \to M$ can be extended to an isometric embedding $g : X \to M$. Urysohn [1927] showed that there is a unique, up to isometry, Polish metric space which is Urysohn. This is what we call *the Urysohn space* and denote it by $\mathbb{U}$. It is also characterized as the unique Polish metric space which is *universal* and *ultrahomogeneous*, where universality means that every Polish metric space can be isometrically embedded into it, and ultrahomogeneity means that any isometry between finite subsets of it can be extended to an isometry of the whole space. Moreover, Katětov [1988] gave a canonical construction of the Urysohn space which, starting from any Polish metric space X, arrives at an extension X^* of X which is isometric to $\mathbb{U}$. This construction has the additional property that, if $\varphi : X \to Y$ is an isometry between Polish metric spaces, then it induces a canonical extension $\varphi^* : X^* \to Y^*$ (which is essentially an isometry of $\mathbb{U}$). These nice properties, together with some careful computations, give a Borel reduction from $\cong_i$ to the orbit equivalence relation on $F(\mathbb{U})$ induced by the application action of $\mathrm{Iso}(\mathbb{U})$, the isometry group of $\mathbb{U}$.

For the other direction of the theorem, it is enough to establish that

$$E^\infty_{\mathrm{Iso}(\mathbb{U})} \leq_B \cong_i,$$

where we recall that E_G^∞ is the universal equivalence relation induced by a Borel action of G. This can be done in three steps.

Step 1. Consider the action of $\mathrm{Iso}(\mathbb{U})$ on $\prod_{n\geq 1} F(\mathbb{U}^n)$ given by

$$\varphi \cdot (R_n) = (\varphi(R_n))$$

and call the corresponding orbit equivalence relation $E^\infty(\mathbb{U})$. One can show that $E^\infty_{\mathrm{Iso}(\mathbb{U})} \leq_B E^\infty(\mathbb{U})$. In fact, it is a general theorem (also proved in (GK)) that the same conclusion remains true if in the above assertion every occurrence of $\mathbb{U}$ is replaced by some arbitrary Polish metric space X.

Step 2. Consider the action of $\mathrm{Iso}(\mathbb{U})$ on $F(\mathbb{U})^{\mathbb{N}}$ given by

$$\varphi \cdot (C_1, C_2, \dots) = (\varphi(C_1), \varphi(C_2), \dots),$$

and let $E^1(\mathbb{U})$ be the corresponding equivalence relation. We argue that $E^\infty(\mathbb{U}) \leq_B E^1(\mathbb{U})$. The argument here can also be interpreted as coding closed relations of arbitrary arity on $\mathbb{U}$ by sequences of unary closed relations (i.e., closed subsets!) of $\mathbb{U}$, while preserving the isometry type of the relations.

It is easy to fabricate examples to see that this cannot be done for general Polish metric spaces. Thus we have to use in an essential way the properties of the Urysohn space. It turns out to be enough (and actually simpler) to argue that there is *some* Polish metric space X such that $E^\infty(\mathbb{U}) \leq_B E^1(X)$, where $E^1(X)$ is defined in a similar fashion as that of $E^1(\mathbb{U})$. The basic idea here is to consider the sequence of isometric embeddings

$$\mathbb{U} \hookrightarrow \mathbb{U}^3 \hookrightarrow \mathbb{U}^9 \hookrightarrow \ldots \hookrightarrow \mathbb{U}^{(3^n)} \hookrightarrow \ldots,$$

where any $\vec{x} \in \mathbb{U}^{(3^n)}$ is identified with $(\vec{x}, \vec{x}, \vec{x}) \in \mathbb{U}^{(3^{n+1})}$ and each $\mathbb{U}^N$, $N \geq 1$, is endowed with the metric

$$d_N(\vec{x}, \vec{y}) = \frac{1}{N} \sum_{i=1}^{N} d(x_i, y_i).$$

Now put $\mathbb{U}^\infty = \bigcup_n \mathbb{U}^{(3^n)}$ and let X be the completion of $\mathbb{U}^\infty$. Then closed relations of arbitrary arity on $\mathbb{U}$ become automatically closed subsets of X. To argue that $E^\infty(\mathbb{U}) \leq_B E^1(X)$, it remains to see that among all isometries of X those induced by isometries of $\mathbb{U}$ are recognizable with the help of a sequence of closed subsets of X. This actually can be done, making use of the universality and homogeneity of $\mathbb{U}$. For full details, see (GK) 2H.

Step 3. It now remains to show that for any Polish metric space X, $E^1(X) \leq_B \cong_i$. Let d be the metric of X, and assume without loss of generality that X has at least 2 elements. Consider then the equivalent metric $\delta = \frac{d}{1+d}$ so that (X, δ) is a Polish metric space and $\delta(x, y) < 1$ for all $x, y \in X$ (in the sequel we also write $\delta < 1$ for short). Given $\vec{C} = (C_0, C_1, \ldots) \in F(X)^{\mathbb{N}}$, consider the Polish metric space $(X_{\vec{C}}, d_{\vec{C}})$ defined as follows: For each nonempty C_n, choose a point $x_n^{\vec{C}}$ not in X and assume that all these points are distinct. Let $X_{\vec{C}} = X \cup \{x_n^{\vec{C}}\}$. Define the metric $d_{\vec{C}}$ as follows: $d_{\vec{C}}$ agrees with δ on X. The distance between any two distinct $x_n^{\vec{C}}, x_m^{\vec{C}}$ is equal to $|n - m| + 1$. Finally, if $u \in X$ we define

$$d_{\vec{C}}(x_n^{\vec{C}}, u) = (n + 2) + \delta(u, C_n),$$

where $\delta(u, C_n)$ is the δ-distance of u from C_n. It is then easy to check that $\vec{C}$ and $\vec{D}$ are $E^1(X)$-equivalent iff $X_{\vec{C}}$ and $X_{\vec{D}}$ are isometric.

We now turn to the method used in (Cl) to prove that for any Borel G-space X, $E_G^X \leq_B \cong_i$. By earlier results of Becker and Kechris [1996] we can assume, without loss of generality, that X is a *compact Polish G-space*, i.e., X is compact and the action of G on X is continuous. The idea then is to study the action of G on X and to code the orbits by constructing Polish metric spaces with distiguished isometry types.

To do this, let d_G be a compatible left-invariant metric on G and d_X be a compatible complete metric on X with $d_G, d_X \leq 1$. Define a new metric d'_G

on G by

$$d'_G(g_1, g_2) = \frac{1}{2} d_G(g_1, g_2) + \frac{1}{2} \sup_{x \in X} d_X(g_1^{-1} \cdot x, g_2^{-1} \cdot x).$$

Then d'_G is still a compatible left-invariant metric on G with $d'_G \leq 1$. In addition, it has the obvious but useful property that for any $x \in X$ and $g_1, g_2 \in G$, $d'_G(g_1, g_2) \geq \frac{1}{2}(g_1^{-1} \cdot x, g_2^{-1} \cdot x)$. Intuitively, the distances of group elements have certain control on the distances of points in a single orbit. Let $\hat{G}$ be the completion of G under d'_G.

Fix a countable dense subset of X, $\{q_n\}$. Let $\pi : \mathbb{Z} \leftrightarrow \mathbb{N}$ be the bijection given by

$$\pi(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -1 - 2n & \text{if } n < 0. \end{cases}$$

Next we define for each $x \in X$ a Polish metric space (M_x, d_x). Let M_x be $\hat{G} \times \mathbb{Z} \times \mathbb{Z}_2$. To define d_x it is enough to specify its values on a convenient dense subset of M_x, namely, $G \times \mathbb{Z} \times \mathbb{Z}_2$. For $g_0, g_1 \in G$, $n_0, n_1 \in \mathbb{Z}$ and $i_0, i_1 \in \mathbb{Z}_2$ let $d_x((g_0, n_0, i_0), (g_1, n_1, i_1)) =$

$$\begin{cases} d'_G(g_0, g_1) & \text{if } i_0 = i_1 \text{ and } n_0 = n_1 \\ \frac{3}{2} + 4^{-|n_0 - n_1|} d'_G(g_0, g_1) & \text{if } i_0 = i_1 \text{ and } |n_0 - n_1| = 1 \\ 1 + 4^{-1-\pi(n_0 - n_1)}[1 + d_X(q_{n_0 - n_1}, g_1^{-1} \cdot x)] & \text{if } i_0 = 0 \text{ and } i_1 = 1 \\ 1 + 4^{-1-\pi(n_1 - n_0)}[1 + d_X(q_{n_1 - n_0}, g_0^{-1} \cdot x)] & \text{if } i_0 = 1 \text{ and } i_1 = 0 \end{cases}$$

One can then check that $xE_G^X y$ iff $(M_x, d_x) \cong_i (M_y, d_y)$ and that the assignment $x \mapsto (M_x, d_x)$ is Borel.

This last method has the advantage that many properties of the group G are carried over to the constructed spaces (M_x, d_x). This allows us to reuse the method in many other situations to give proofs of lower bounds on the complexity of classification problems.

§3. The classification of special classes of Polish metric spaces. The main result of §2 settles the question concerning the classification for general Polish metric spaces. It is however of further interest to understand the complexity of the isometry problem for special classes of Polish metric spaces. We shall consider three groups of properties used in defining the special classes in our consideration. These are (in some very general sense) connectedness properties, properties of the isometry groups (like homogeneity, rigidity, etc.), and compactness properties.

3.A. Ultrametric and zero-dimensional spaces. A metric d on a space X is an *ultrametric* if for any $x, y, z \in X$,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Because of the continuity of the metric function to check that d is an ultrametric, it is enough to verify this inequality only on a countable dense set of points in X. It follows that the class of all ultrametric Polish spaces is a Borel subset of the standard Borel space of all Polish metric spaces $F(\mathbb{U})$, and thus is itself standard Borel. For the classification of this class of spaces we have the following characterization for its complexity.

THEOREM 3.1 (GK). *The isometry equivalence relation of ultrametric Polish spaces is Borel bireducible with graph isomorphism.*

Thus the classification of ultrametric Polish spaces is strictly simpler than that of all Polish metric spaces. Note that every ultrametric space is 0-dimensional. (Recall that a space is *0-dimensional* if it has a clopen basis.) Therefore we have a lower bound for the isometry equivalence relation of 0-dimensional Polish metric spaces, namely graph isomorphism. This implies, in particular, that this equivalence relation is non-Borel.

Making use of Hjorth's theory of turbulence (see Hjorth [2000]) we can obtain the following result.

THEOREM 3.2 (Cl). *The isometry equivalence relation of* 0*-dimensional Polish metric spaces is not Borel reducible to graph isomorphism, thus it is strictly bigger, in the sense of Borel reducibility, than graph isomorphism.*

However, we do not know the exact complexity of this classification problem. Solecki noted that the isometry equivalence relation of 1-dimensional Polish metric spaces is also not Borel reducible to graph isomorphism, by essentially the same proof as for the above theorem.

Note also that 0-dimensionality and 1-dimensionality are not Borel properties; they are $\mathbf{\Sigma}^1_1$.

In contrast to 0-dimensionality one can consider connectedness. It is not hard to see that graph isomorphism is Borel reducible to the isometry equivalence relation of connected Polish metric spaces. But the exact complexity of this equivalence relation is not known.

3.B. Homogeneous, ultrahomogeneous and rigid spaces. A metric space X is *homogeneous* if its group of isometries, $\mathrm{Iso}(X)$, acts on X transitively, i.e., for any $x, y \in X$ there is an isometry $\varphi \in \mathrm{Iso}(X)$ so that $\varphi(x) = y$. A homogeneous space is intuitively regarded as having a large isometry group. We first consider discrete spaces, i.e., those with a discrete topology.

THEOREM 3.3 (Cl). *The isometry equivalence relation of homogeneous discrete Polish metric spaces is Borel bireducible with graph isomorphism.*

About general homogeneous spaces we have the following partial result.

THEOREM 3.4 (Cl). *If G is an abelian Polish group, then E_G^∞ is Borel reducible to the isometry equivalence relation of homogeneous Polish metric spaces.*

Consequently, again by Hjorth's theory of turbulence, this isometry equivalence relation is not Borel reducible to graph isomorphism.

In the other extreme, we can consider the class of rigid Polish metric spaces, i.e., those spaces with no non-trivial isometries (thus their isometry group is the smallest possible).

THEOREM 3.5. *If G is a Polish group, X a Borel G-space and the action is free, then E_G^X is Borel reducible to the isometry equivalence relation of rigid Polish metric spaces. In particular, the isometry equivalence relation of rigid Polish metric spaces is not Borel reducible to graph isomorphism.*

Note that E_G^X is Borel when G acts freely on X, but there are free actions whose orbit equivalence relation is of arbitrarily high Borel complexity. Thus the isometry of rigid Polish metric spaces is not a Borel equivalence relation. However, the space of rigid Polish metric spaces can be seen to be $\mathbf{\Pi}_1^1$, thus the isometry may as well be a $\mathbf{\Delta}_1^1$ equivalence relation on a $\mathbf{\Pi}_1^1$ subset of a standard Borel space.

Recall from §2 (where we discuss the Urysohn space) that a metric space X is *ultrahomogeneous* if any isometry between two finite subsets of X can be extended to an isometry of the whole X. Ultrahomogeneous spaces are necessarily homogeneous. Here using the proofs of Theorem 3.2 and 3.4 we get the following result.

THEOREM 3.6 (Cl). *The isometry equivalence relation of ultrahomogeneous Polish metric spaces is not Borel reducible to graph isomorphism.*

As in the case of homogeneous spaces, a complete result can be obtained if we restrict our attention to discrete spaces. Let F_2 be the equivalence relation of equality of countable sets of reals, i.e., for $(x_n), (y_n) \in \mathbb{R}^{\mathbb{N}}$,

$$(x_n)F_2(y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}.$$

This is a Borel equivalence relation strictly simpler than graph isomorphism.

THEOREM 3.7 (Cl). *The isometry equivalence relation of ultrahomogeneous discrete Polish metric spaces is Borel bireducible with F_2.*

Finally, in the special case of ultrametric spaces we have the following result.

THEOREM 3.8 (GK). *The isometry equivalence relation of ultrahomogeneous discrete ultrametric Polish spaces is Borel bireducible with F_2. The same is true for homogeneous ultrametric Polish spaces.*

3.C. Locally compact spaces. First there is the following result of Gromov.

THEOREM 3.9 (Gromov [1999] $3.11\frac{1}{2}_+$ or 3.27). *The isometry equivalence relation of compact Polish metric spaces is concretely classifiable.*

The next step along these lines would be to calculate the complexity of isometry on locally compact Polish metric spaces. It is not hard to see that isometry of general discrete Polish metric spaces has exactly the same complexity as graph isomorphism. Therefore isometry of locally compact Polish metric spaces is at least as complex as graph isomorphism. The crucial question here is whether it is exactly the same as graph isomorphism.

Our results on the isometry groups of such spaces, which we will discuss in §4, led us to the conjecture that in fact isometry of locally compact Polish metric spaces is Borel reducible to graph isomorphism, and therefore it has exactly the same complexity as graph isomorphism. Hjorth has shown that a weaker form of this conjecture is in fact true, namely that isometry of locally compact Polish spaces is reducible by a provably $\mathbf{\Delta}^1_2$ function to graph isomorphism. This provides strong evidence for the truth of the conjecture.

One can look further at important subclasses of locally compact spaces. These subclasses are again defined by the properties we have encountered in earlier parts of this section. Here we have several complete results.

THEOREM 3.10 (GK). *The isometry equivalence relation of* 0*-dimensional locally compact Polish metric spaces is Borel bireducible with graph isomorphism.*

At the other extreme are the connected locally compact spaces. The universal countable Borel equivalence relation, E_∞, comes into play at this point. Note that the complexity of E_∞ is rather mild compared to other complex equivalence relations we have seen in earlier parts of this paper. For example, it is well known that $E_\infty <_B F_2$. From the realization of E_∞ as the isomorphism relation among locally finite trees, it is not hard to see that E_∞ is Borel reducible to the isometry of connected locally compact Polish metric spaces. Again results on their isometry groups motivated our conjecture that the isometry of connected locally compact Polish metric spaces is Borel bireducible with E_∞. This has been confirmed by Hjorth.

THEOREM 3.11 (Hjorth, see (GK)). *The isometry equivalence relation of connected locally compact Polish metric spaces is Borel bireducible with* E_∞.

In fact this is only a part of a general result Hjorth proved. In order to state this general result, we need to elaborate on some concepts and techniques developed in (GK).

Assume that (X, d) is a locally compact Polish metric space. For each $x \in X$, we define its *radius of compactness*, denoted by $\rho(x)$, to be the supremum of $r > 0$ such that the closed ball of radius r with center x is compact. $\rho(x)$ is always positive since X is locally compact. If $\rho(x) = \infty$ for some x (equivalently for all x), then d is called a *Heine-Borel* metric and (X, d) a Heine-Borel space. Now define an equivalence relation E on X by letting xEy when either $x = y$ or the following happens:

> for some $u_0 = x, u_1, \dots, u_n = y$ we have that for all $i < n$, $u_i \neq u_{i+1}$ and $d(u_i, u_{i+1}) < \rho(u_i)$; and, conversely, for some $v_0 = y, v_1, \dots, v_m = x$ we have that for all $j < m$, $v_j \neq v_{j+1}$ and $d(v_j, v_{j+1}) < \rho(v_j)$.

We then call the E-equivalence class of $x \in X$ the *pseudo-component* of x, and we call X *pseudo-connected* if it has only one pseudo-component.

It is easy to check that each pseudo-component is clopen, therefore there are only countably many pseudo-components in each locally compact Polish metric space. It also follows that connected locally compact Polish metric spaces are pseudo-connected. But pseudo-connectedness certainly goes beyond connectedness, since it is trivial from the definition that all Heine-Borel spaces are pseudo-connected.

What we actually conjectured, and Hjorth proved, was that the isometry equivalence relation of pseudo-connected locally compact Polish metric spaces is Borel bireducible with E_∞. Moreover, the isometry of locally compact Polish metric spaces with finitely many pseudo-components has exactly the same complexity.

Pseudo-connectedness proves to be a key concept in understanding the structure of locally compact Polish metric spaces and their isometry groups. It also generates new subclasses of Polish metric spaces the complexity of whose classification problems seems natural. An example of such results is the following.

THEOREM 3.12 (GK). *The isometry of homogeneous pseudo-connected locally compact Polish metric spaces is concretely classifiable.*

It is proved in (Cl) that the isometry of homogeneous locally compact Polish metric spaces with two pseudo-components is no longer concretely classifiable. In fact, Louveau showed the following stronger result using the method of (Cl).

THEOREM 3.13. *If G is an abelian and countable Polish group, then E_G^∞ is Borel reducible to the isometry of ultrahomogeneous locally compact Polish metric spaces with two pseudo-components.*

Using a similar proof as that of Theorem 3.5 one can show that, for a countable Polish group G and a free action of G on a Borel G-space X, E_G^X is Borel reducible to the isometry of rigid locally compact Polish metric spaces with two pseudo-components.

Finally, let E_0 be the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0y \Leftrightarrow \exists n \forall m \geq n\,(x_m = y_m).$$

This is a countable Borel equivalence relation strictly simpler than E_∞. In fact it is the simplest non-concretely classifiable equivalence relation. Then we have

THEOREM 3.14 (GK). *The isometry of Heine-Borel ultrametric Polish spaces is Borel bireducible with E_0.*

§4. Groups of isometries.

It turns out that our work on the classification problems has some interesting applications to the study of isometries of various metric spaces. For a Polish metric space X, we denote by $\mathrm{Iso}(X)$ the group of isometries of X, where the group operation is composition. Equipped with the pointwise convergence topology, $\mathrm{Iso}(X)$ becomes a Polish group. Our first result shows that the converse is true, thus completely characterizes the isometry groups of Polish metric spaces.

THEOREM 4.1 (GK). *Up to (topological group) isomorphism the isometry groups of Polish metric spaces are exactly the Polish groups.*

The proof of this theorem closely relates to the proof of Theorem 2.1 in (GK), which we sketched in §2, and makes essential use of the Urysohn space.

We next consider the case of locally compact separable metric spaces (X, d), where d is not necessarily complete. For any such space it still turns out that its isometry group is Polish. Moreover, the analysis of pseudo-components we sketched in §4 still goes through for such spaces. In case the space is pseudo-connected, we have the following result for its isometry group.

THEOREM 4.2 (GK). *Let X be a pseudo-connected locally compact separable metric space. Then $Iso(X)$ is locally compact.*

This generalizes earlier results of van Dantzig and van der Waerden [1928] (for the connected case) and Strantzalos [1989] (see also Strantzalos [1974] and Manoussos and Strantzalos [2000]).

Using this, and some further constructions, we can characterize completely the isometry groups of locally compact separable metric spaces.

THEOREM 4.3 (GK). *Up to (topological group) isomorphism the following classes of of groups are the same:*

i) *The isometry groups of locally compact separable metric spaces.*
ii) *The isometry groups of locally compact Polish metric spaces.*
iii) *The isometry groups of σ-compact Polish metric spaces.*
iv) *The closed subgroups of groups of the form*

$$\prod_n (S_\infty \ltimes G_n^{\mathbb{N}}),$$

where (G_n) is a sequence of locally compact Polish groups, and $S_\infty \ltimes G^{\mathbb{N}}$ is the semi-direct product of S_∞ and $G^{\mathbb{N}}$, with S_∞ acting on $G^{\mathbb{N}}$ by $g \cdot x(i) = x(g^{-1}(i))$.

This characterization can in turn be used to get information about actions of such isometry groups.

THEOREM 4.4 (GK). *Let X be a locally compact separable metric space and Y be a Borel $Iso(X)$-space. Then $E^Y_{Iso(X)}$ is Borel reducible to graph isomorphism.*

This extends a result of Hjorth [2000] who proved such a theorem for countable products of locally compact Polish groups.

These theorems helped to motivate our conjectures discussed in §3.C, since it seems that the complexity of a classification problem is connected to the automorphism groups of the objects and their actions.

It is also shown in (GK) that, up to isomorphism, the isometry groups of 0-dimensional locally compact Polish metric spaces are exactly the closed subgroups of S_∞. Concerning ultrametric Polish spaces it is not hard to see that these isometry groups are, up to isomorphism, closed subgroups of S_∞ but we do not know how to characterize them.

§5. Open problems. In this final section, we discuss some open problems and directions for further research. We are not attempting to compile a complete list of open questions that arise here, some of which we mentioned earlier in this paper.

One important question left open from our study is the classification of locally compact Polish metric spaces.

PROBLEM 5.1. Determine the exact complexity of the isometry of locally compact Polish metric spaces.

The problem seems quite challenging despite Hjorth's result mentioned in §3.C, following Theorem 3.8. The difficulty lies in trying to understand the interactions of various pseudo-components of the space, when infintely many of them are present.

Another interesting problem is the classification of compact Polish spaces up to homeomorphism.

PROBLEM 5.2. Determine the exact complexity of homeomorphism of compact metric spaces.

Hjorth [2000] has shown that this equivalence relation is strictly above graph isomorphism. It has been also proved (by Kechris and Solecki) that homeomorphism of compact Polish spaces is Borel reducible to an equivalence relation induced by a Borel action of a Polish group. In fact, the Banach-Stone Theorem (see, e.g., Semadeni [1971], 7.8.4) implies that, for compact Polish spaces X and Y,

$$\begin{aligned}&X \text{ and } Y \text{ are homeomorphic}\\ &\quad\Leftrightarrow C(X) \text{ and } C(Y) \text{ are isometric Banach spaces}\\ &\quad\Leftrightarrow C(X) \text{ and } C(Y) \text{ are isometric as Polish spaces,}\end{aligned}$$

where $C(X)$ denotes the space of all continuous functions from X into $\mathbb{R}$ with the supnorm (metric). It is easy to see that the map $X \mapsto C(X)$ is Borel. Thus by our Theorem 2.1, we have an alternative proof of the fact that homeomorphism of compact Polish spaces is Borel reducible to an orbit equivalence relation induced by a Borel action of a Polish group.

We also see from the above computation that linear isometry of Banach spacs is above homeomorphism of compact metric spaces. This leads to the following problem.

PROBLEM 5.3. Determine the exact complexity of isomorphism and linear isometry of separable Banach spaces.

We believe that the study of these problems will further our understanding of the mathematical structures in question, as well as that of the descriptive set theory of definable equivalence relations.

REFERENCES

H. BECKER AND A. S. KECHRIS [1996], ***The descriptive set theory of Polish group actions***, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press.

J. D. CLEMENS [2001], ***Ph.D. thesis***, University of California, Berkeley.

R. DOUGHERTY, S. JACKSON, AND A.S. KECHRIS [1994], *The structure of hyperfinite Borel equivalence relations*, ***Transactions of the American Mathematical Society***, vol. 341, no. 1, pp. 193–225.

S. GAO AND A. S. KECHRIS [2000], *On the classification of Polish metric spaces up to isometry*, preprint.

M. GROMOV [1999], ***Metric structures for Riemannian and non-Riemannian spaces***, Progress in Mathematics, vol. 152, Birkhäuser.

G. HJORTH [2000], ***Classification and orbit equivalence relations***, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society.

G. HJORTH AND A.S. KECHRIS [1996], *Borel equivalence relations and classifications of countable models*, ***Annals of Pure and Applied Logic***, vol. 82, pp. 221–272.

M. KATĚTOV [1988], *On universal metric spaces*, ***General topology and its relations to modern analysis and algebra, VI (Prague, 1986)***, Research and Exposition in Mathematics, vol. 16, Heldermann, Berlin, pp. 323–330.

A. S. KECHRIS [1995], ***Classical descriptive set theory***, Springer-Verlag.

A. S. KECHRIS [1999], *New directions in descriptive set theory*, this BULLETIN, vol. 5, no. 2, pp. 161–179.

A. S. KECHRIS [2001], *Actions of Polish groups and classification problems*, ***Analysis and logic***, London Mathematical Society Lecture Notes Series, Cambridge University Press, to appear.

A. MANOUSSOS AND P. STRANTZALOS [2000], *The role of connectedness in the structure and the action of groups of isometries of locally compact metric spaces*, preprint.

Z. SEMADENI [1971], ***Banach spaces of continuous functions***, PWN Polish Scientific Publishers.

P. STRANTZALOS [1974], *Dynamische Systeme und Topologische Aktionen*, ***Manuscripta Mathematica***, vol. 13, pp. 207–211.

P. STRANTZALOS [1989], *Action by isometries*, ***Transformation groups (Osaka 1987)***, Lecture Notes in Mathematics, vol. 1375, Springer, pp. 319–325.

P. URYSOHN [1927], *Sur en espace métrique universel*, ***Bulletin des Sciences Mathématiques***, vol. 51, pp. 43–64, 74–90.

V. V. USPENSKIĬ [1986], *A universal topological group with a countable basis*, ***Functional Analysis and Its Applications***, vol. 20, pp. 86–87.

V. V. USPENSKIĬ [1990], *On the group of isometries of the Urysohn universal metric space*, ***Commentationes Mathematicae Universitatis Carolinae***, vol. 31, no. 1, pp. 181–182.

D. van Dantzig and B.L. van der Waerden [1928], *Über metrisch homogene Räume*, ***Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg***, vol. 6, pp. 374–376.

A. M. Vershik [1998], *The universal Urysohn space, Gromov metric triples and random metrics on the natural numbers*, ***Russian Mathematical Surveys***, vol. 53, no. 5, pp. 921–928.

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