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FULL FROBENIUS GROUPS OF FINITE MORLEY RANK AND THE FEIT-THOMPSON THEOREM

ERIC JALIGOT

Abstract. We show how the notion of full Frobenius group of finite Morley rank generalizes that of bad group, and how it seems to be more appropriate when we consider the possible existence (still unknown) of nonalgebraic simple groups of finite Morley rank of a certain type, notably with no involution. We also show how these groups appear as a major obstacle in the analysis of *FT*-groups, if one tries to extend the Feit-Thompson theorem to groups of finite Morley rank.

§1. Introduction. The study of stability in model theory begins when M. Morley proved his famous theorem about first-order theories categorical in uncountable cardinalities [31]. It appeared then that a dimension was attached to such theories, a kind of Cantor-Bendixon rank on the definable subsets of a model, now called the *Morley rank*. It turns out that, in algebraic geometry over an algebraically closed field K, the Zariski dimension of a constructible set in some K^n coincides with its (finite) Morley rank, and that algebraic groups over such a field are of finite Morley rank.

A question about groups of finite Morley rank is, as has been the case for finite groups for a long time, the classification of the simple ones. Such a group is conjecturally—this is known as the *Cherlin-Zil'ber conjecture*—algebraic over an algebraically closed field K, that is essentially a Zariski closed group of matrices with entries in K.

1.1. Some history. In a group G of finite Morley rank, strictly descending chains of definable subgroups are finite. In particular G has a smallest definable subgroup G° of finite index, which is normal and called the *connected component*.

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A cornerstone in the analysis of simple groups of finite Morley rank, which gave some hopes in the eighties to give a positive answer to the Cherlin-Zil'ber conjecture, is the Field Theorem of B. Zil'ber [14, 35, Théorème 3.7]. It says that a solvable nonnilpotent connected group of finite Morley rank G interprets an algebraically closed field K in such a way that a *definable section* of G (that is a group H/L where $L \trianglelefteq H \leq G$ are two definable subgroups) is isomorphic to the additive group K^+ and another such section is isomorphic to an infinite definable subgroup T of the multiplicative group $K^\times$.

Furthermore, fields of finite Morley rank are algebraically closed by a well-known theorem of A. Macintyre [30].

As a warm-up exercise to the Field Theorem, it is fairly easy to interpret the field K in the pure group of upper-triangular two-by-two matrices

$$G = \left\{ \begin{pmatrix} 1 & u \\ 0 & t \end{pmatrix} : u \in K^+, t \in K^\times \right\},$$

where K is a given field. The subgroup of unipotent matrices and the torus carry the structure of the additive group K^+ and the multiplicative group $K^\times$ respectively; these subgroups are both definable as they are the centralizers in G of their nontrivial elements.

It was proved ([11]) in the late eighties that, in a group of finite Morley rank, Sylow 2-subgroups are conjugate and have a well-determined structure. They are essentially a product of two subgroups: a 2-*unipotent* subgroup and a 2-*torus*, each having the isomorphism type of Sylow 2-subgroups of an algebraic group over an algebraically closed field of characteristic two and different from two respectively.

Since that time, this Sylow theory for the prime 2 has been the key for a classification program of simple groups of finite Morley rank, proposed notably by A. Borovik and exposed in [10] (see also the list of open problems at the end of [14]). This program is inspired by the classification of finite simple groups, or more precisely the 'second generation' of classification, that is a 'revisionist' approach with inductive assumptions. Some results will be stated in section 2 below.

The idea is to recognize a simple group starting from properties of its subgroups, and thus to do some *local analysis*. It is rather easier to work with *tame* groups of finite Morley rank, avoiding two kinds of pathologies whose existence is still unknown: *bad groups* and *bad fields*.

A bad group is a nonsolvable connected group of finite Morley rank in which proper, definable, and connected subgroups are nilpotent (so that one cannot use, *a priori*, the Field Theorem). A bad field is a field K of finite Morley rank with a proper, infinite, and definable subgroup T of the multiplicative group $K^\times$. A group of finite Morley rank is said to be tame if it has no definable bad sections and if it interprets no bad fields. A tame group is, roughly, a group in which Borel subgroups[1] look like those of algebraic groups, by the Field Theorem.

In a simple tame (infinite) group, Sylow 2-subgroups are automatically infinite: it suffices to apply the Field Theorem in a Borel subgroup B, and then to 'lift'[2] the Sylow 2-subgroups in B (see also [14, Theorem B.1]). In what follows we will look at groups in which the preceding argument cannot be done.

1.2. Challenging problems. It is known from [15, 17] that, in a simple bad group G, the (nilpotent) Borel subgroups are conjugate and if B is one of them, then B intersects B^g trivially for each element g of $G \setminus B$. Furthermore, G is covered by the conjugates of B. In the terminology adopted here, we shall say that $B < G$ is a *full Frobenius group* (see section 3 below). In particular, such a group has no involution and would be a radical counterexample to the Cherlin-Zil'ber conjecture.

On the other hand, many people now believe that bad fields exist, at least in characteristic zero, even though the fields with a multiplicative subgroup built by B. Poizat in [34] have infinite rank. And it seems that the existence of such fields would not contradict the Cherlin-Zil'ber conjecture: they only allow one to build, *a priori*, nonalgebraic solvable groups.

If bad fields exist, it is natural to try to extend the Feit-Thompson theorem [19] on the solvability of finite groups of odd order, the question being here:

QUESTION 1.1. Does a simple infinite group of finite Morley rank necessarily have infinite Sylow 2-subgroups?

An *FT-group* is a nonsolvable connected group of finite Morley rank whose proper, definable, and connected subgroups are solvable. This class of groups which contains the hypothetical bad groups is nonempty: groups of the form $\mathrm{PSL}_2(K)$ for some algebraically closed field K are (exactly) the simple algebraic groups with this property.

A minimal counterexample to question 1.1 is also a simple FT-group (in which Borel subgroups have trivial Sylow 2-subgroups). In this paper we shall prove the following theorem:

THEOREM 1.2. *Let G be a simple FT-group. Then either $B < G$ is a full Frobenius group for every Borel subgroup B (and in this case Borel subgroups are conjugate), or G contains a nonisolated Borel subgroup.*

A Borel subgroup is *isolated* if it has too many finite intersections with other Borel subgroups (see section 4 below). In $\mathrm{PSL}_2(K)$, there are no isolated Borel subgroups because two Borel subgroups intersect at least in

[1] A Borel subgroup of a group of finite Morley rank is a definable, connected, and solvable subgroup which is maximal with respect to these properties.

[2] A Sylow p-subgroup of a definable section H/L of a solvable group B of finite Morley rank is the image of a Sylow p-subgoup of the group H [6]. For Sylow 2-subgroups, this is true even if B is nonsolvable [36].

an infinite torus. The presence of a nonisolated Borel subgroup in the second case of theorem 1.2 implies in particular that some intersections are infinite between distinct Borel subgroups.

The notion of FT-group is crucial for the classification of simple groups of finite Morley rank. They appear as the most critical configurations that have to be studied, both in the case where Sylow 2-subgroups are assumed to be infinite (see section 2 below) and in the case where they are assumed to be finite.

We should (now) say that theorem 1.2 has no revolutionary character; our aim is simply to give a more general way of looking at bad groups, in connection with the possible existence of bad fields.

When they analysed a minimal counterexample to their theorem, W. Feit and J. Thompson looked at a simple group of odd order where proper subgroups were solvable, which may be viewed as a finite analog of a minimal counterexample to question 1.1—the 'lyonnaise' terminology 'FT-group' comes from this analogy. It should be noted at this point that analogies between groups of finite Morley rank and finite groups are generally visible when one works with connected components in groups of finite Morley rank.

Theorem 1.2 splits question 1.1 in the two following:

QUESTION 1.3. Are there full Frobenius groups of finite Morley rank?

This question seems to be, in general, as hard as the question of the existence of bad groups.

QUESTION 1.4. Is there a simple FT-group with a nonisolated Borel subgroup and with finite Sylow 2-subgroups?

As there are, in the configurations of question 1.4, some infinite intersections between distinct Borel subgroups, there are some hopes to adapt, at some point, some arguments of the Feit-Thompson theorem. Some kind of local analysis in the style of [8] could be expected in order to get some contradictions, or at least to have some good descriptions of the groups considered. These hopes essentially come from the fact that the structure of connected solvable groups of finite Morley rank is now very well known by the work of O. Frécon [20].

All our notations and standard definitions are those of [14]. In this paper we only deal with groups of finite Morley rank, so that arguments remain easy in this unified context. But most of the ideas and arguments presented here could probably be adapted in larger classes of groups, notably the stable groups with the condition $\mathfrak{N}$ of [39].

§2. FT-groups and Sylow 2-subgroups.

An *involution* in a group is an element of order 2. It is shown in [11] that Sylow 2-subgroups are conjugate in a group G of finite Morley rank and if S is one of them, then its connected component S°—defined by $S \cap d(S)^{\circ}$ where $d(S)$ denotes the smallest definable subgroup containing S, as it is not definable in general—is the central product $B * T$ of a definable subgroup B of bounded exponent (a 2-*unipotent* subgroup) and a divisible abelian subgroup T (a 2-*torus*). The following terminologies come from the type of Sylow 2-subgroups in algebraic groups, depending on whether the underlying field is of characteristic 2 or not. G is said to be:

- of *degenerate* type if $B = T = 1$, that is if S is finite,
- of *even* type if $B \neq 1$ and $T = 1$,
- of *odd* type if $B = 1$ and $T \neq 1$,
- of *mixed* type if $B \neq 1$ and $T \neq 1$.

A simple FT-group is in particular a simple K^*-*group*, the latter terminology being the one used in the classification of all potential minimal counterexamples to the Cherlin-Zil'ber conjecture [27, 25, 5, 2, 13, 12, 1, 9]. Recall from [10] that a K-*group* is a group of finite Morley rank in which every definable, infinite, and simple section is algebraic over an algebraically closed field, and that a K^*-*group* is an infinite group of finite Morley rank in which every proper and definable subgroup is a K-group. A minimal counterexample to the Cherlin-Zil'ber conjecture is of course a simple K^*-group.

It turns out that FT-groups are probably the most critical configurations in the classification of simple K^*-groups, but it seems conjecturally reasonable to assert that the notion should characterize $\mathrm{PSL}_2(K)$, provided Sylow 2-subgroups are infinite.

What can be said about FT-groups according to the nature of their Sylow 2-subgroups? The following fact is proved in [27] for simple K^*-groups; the proof for FT-groups can also easily be extracted from the original paper [3] which dealt with tame K^*-groups of mixed type.

Fact 2.1. A simple FT-group cannot be of mixed type.

What about groups of even type? For the following fact, proved in [25], recall from [4] that a proper definable subgroup M of a group of finite Morley rank G is said to be *weakly embedded* in G if it has infinite Sylow 2-subgroups and if $M \cap M^g$ has finite Sylow 2-subgroups for every $g \in G \setminus M$.

Fact 2.2. A simple K^*-group of even type and with a weakly embedded subgroup is isomorphic to $\mathrm{PSL}_2(K)$ for some algebraically closed field K of characteristic 2.

In the paper [4], which dealt with tame K^*-groups of even type and with a weakly embedded subgroup, it is shown in corollary 10.2 that a simple FT-group of even type contains a weakly embedded subgroup. Thus, it follows from fact 2.2 that a simple FT-group of even type is isomorphic to $\mathrm{PSL}_2(K)$ for some algebraically closed field K of characteristic 2.

Fact 2.2 is a cornerstone in the classification of simple K^*-groups of even type (in progress, see [1]) without the 'tameness' assumption.

The two preceding facts show that FT-groups have to be dealt with only if they are of odd or degenerate type. We wish now to mention a lemma ([28, lemme 2.14]) proved for the study of simple FT-groups of odd type. It can be seen as a generalization of the fact that unipotent subgroups of $\mathrm{PSL}_2(K)$ are selfcentralizing; note also that there is no assumption about Sylow 2-subgroups.

FACT 2.3. Let G be a simple FT-group, B a Borel subgroup of G, and U a B-minimal subgroup of B (that is an infinite normal subgroup of B, minimal with respect to these properties). Then $C_G(u)^\circ = C_B(U)^\circ$ for every nontrivial element u of U.

It was shown in [28] that a simple FT-group of odd type is isomorphic to $\mathrm{PSL}_2(K)$ for some algebraically closed field K of characteristic different from 2 under additional assumptions on the centralizers of involutions:

FACT 2.4. Let G be a simple FT-group of odd type, S a Sylow 2-subgroup of G, and i an involution of S°. Assume that there is a Borel subgroup B of G containing $C_G(i)^\circ$, and a B-minimal subgroup U of B such that $C_{C_G(i)^\circ}(U)$ is finite. Then G is isomorphic to $PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.

In contrast, we shall now consider groups without involutions or with finite Sylow 2-subgroups.

§3. Full Frobenius groups. A group G with a proper subgroup C is said to be a *Frobenius group* if the only elements g of G such that $C \cap C^g$ is nontrivial are those of C. In this case we denote it by $C < G$ and C is said to be a *Frobenius complement*; this comes from the fact that if G is finite, then G splits as $N \rtimes C$ for some normal subgroup N by a well-known theorem of G. Frobenius (see [37, Chapter 6, Theorem 2.2]). We keep this usual terminology of complement, though the Frobenius complements we shall study now have no supplement N as in the finite case! In fact, we consider Frobenius groups $C < G$ such that $G = \bigcup_{g \in G} C^g$, and we say in this case that they are *full*.

Note first, that the preceding definition is not useless as is shown in combinatorial group theory. The following striking fact, stated as theorem 17 in [24][3], shows that full Frobenius groups do exist with a large supply of (given) Frobenius complements.

FACT 3.1. Let C be a finite or countable group without involutions and containing an element of order n big enough (e.g., $n > 10^{12}$ or n infinite).

[3]The theorem is only stated in [24] and also in [23]. The reference given for the proof is [22], which seems to remain unpublished and has not been seen by the author of the present paper.

Then C embeds into a simple and 2-generated group G such that $C < G$ is a full Frobenius group.

Frobenius groups of finite Morley rank must be split if the Cherlin-Zil'ber conjecture is true [18]. For the splitting properties in the context of groups of finite Morley rank, the reader can also consult [33].

We now list a few remarks about full Frobenius groups:

REMARK 3.2. If $C < G$ is a full Frobenius group, then:

(i) a simple counting argument shows (without use of the Frobenius theorem) that G is infinite. For if g and c were the cardinals of G and C respectively, then $g = (c-1) \times (g/c) + 1 = g - g/c + 1$ and $g = c$.

(ii) G is 'semisimple': its abelian normal subgroups are trivial.

(iii) G has no involution: for if i were an involution in C, $g \in G \setminus C$, and $j = i^g$, then i, ij, and j would be in the same conjugate of C as $(ij)^i = (ij)^{-1} = (ij)^j$.

If the group G is of finite Morley rank, then one can prove the following generalities:

PROPOSITION 3.3. *Let $C < G$ be a full Frobenius group, such that G is of finite Morley rank and connected. Then*:

(i) *$C_G(c) = C_C(c)$ is infinite for each nontrivial element c of C.*

(ii) *C is definable in the pure group G and is connected.*

(iii) *$CgC \cap Cg^{-1}C = \emptyset$ for each element g of $G \setminus C$.*

(iv) $\mathrm{rk}(G) \geq 2\,\mathrm{rk}(C) + 1$ *where* rk *denotes the Morley rank.*

PROOF. Suppose that an element $c \in C$ has a finite centralizer in G. Then its conjugacy class as well as the conjugacy class of c^{-1} are generic subsets of G. The latter being connected, it cannot contain two disjoint generic subsets [16]. So there is some $g \in G$ such that $c^g = c^{-1}$, and $C_G(c)$ is of index 2 in the subgroup of elements which centralize or invert c. The latter is thus finite and must contain an involution, which is impossible and proves the first point.

For the second point, note that $\langle\, C_G(c)^\circ : c \in C \setminus \{1\}\,\rangle$ is a nontrivial normal subgroup of C. As it is definable by Zil'ber's Indecomposability theorem [40], its normalizer, which is exactly C, is also definable in the pure group G. Then C must also be connected as G is itself connected and covered by disjoint copies of C. The second point is proved.

Let now $g \in G \setminus C$. If $cgc' = g^{-1}$ for two elements c and c' of C, then $(cg)^2$ is a nontrivial element of C; as it is centralized by cg, then cg and g must be in C, which is impossible and proves the third point.

Finally $\mathrm{rk}(CgC) = 2\,\mathrm{rk}(C)$ as C intersects trivially C^g. Thus $\mathrm{rk}(G) \geq 2\,\mathrm{rk}(C)$ and the previous point together with the connectedness of G shows that the inequality is strict. ⊣

Note that equality is realized in (iv) if a bad group of rank 3 of [16] exists. We do not know if a group G as in the preceding proposition must be simple; we can only prove the following:

PROPOSITION 3.4. *Let $C < G$ be a full Frobenius group, such that G is of finite Morley rank and connected. Then there exists a nontrivial definable simple subgroup G_1 of G such that $(C \cap G_1) < G_1$ is a full Frobenius group.*

PROOF. Let G and C be as in the statement. First, we claim that if G_1 is a definable connected nontrivial subgroup of G, then G_1 is a full Frobenius group. To see this, note that $G_1 = \bigcup_{g \in G}(G_1 \cap C^g)$ and if we denote by Γ_g the nontrivial intersections, then each $\Gamma_g < G_1$ is a Frobenius group. In particular one can easily see that the conjugacy class of Γ_g is generic in G_1:

$$\mathrm{rk}\left(\bigcup_{l \in G_1} {\Gamma_g}^l\right) = \mathrm{rk}(\Gamma_g) + \mathrm{rk}(G_1/N_{G_1}(\Gamma_g)) = \mathrm{rk}(G_1).$$

As the latter is assumed to be connected, these subgroups Γ_g are conjugate in G_1, proving the claim.

To prove the proposition now, take G_1 to be a nontrivial definable subgroup which is not contained in a conjugate of C, and minimal with all these properties. Then G_1 is not abelian and is definably simple as otherwise a definable proper nontrivial subgroup would be contained in a conjugate of C by minimality, and its normalizer would also be contained in this conjugate. But it is a well-known corollary of Zil'ber's Indecomposability theorem that definable simplicity is equivalent to simplicity for nonabelian groups of finite Morley rank. Thus G_1 is simple. One can suppose that it has a nontrivial intersection with C by taking a conjugate if necessary. ⊣

Let us finish this section with a remark: if one tries to construct a simple full Frobenius group of finite Morley rank, then the preceding proposition shows that one can forget simplicity in the construction!

§4. Isolated Borel subgroups. We say that a Borel subgroup B of a group of finite Morley rank G is *isolated* if for every element g of $G \setminus N_G(B)$ and every Borel subgroup K of G, $B \cap K$ is finite or $B^g \cap K$ is finite. As an immediate consequence of this definition, any two distinct conjugates of an isolated Borel subgroup B have a finite intersection.

We will occasionally use without reference the well-known fact that every element of a nontrivial connected solvable group G of finite Morley rank has an infinite centralizer[4]. For if $g \in G$ had a finite centralizer, its conjugacy class as well as $[g, G]$ and G' would be generic in G. But it is a well-known corollary of Zil'ber's Indecomposability theorem [14, 35] that the

[4]The question of [35], whether an element can have finite centralizer in a connected group of finite Morley rank, remains an open problem in general. Notice simply that proposition 3.3 gives a negative answer for full Frobenius groups.

derived subgroup G' of a connected group G is definable and connected, thus $G = G'$, which yields a contradiction as G is solvable[5].

LEMMA 4.1. *Let G be a simple FT-group which contains an isolated Borel subgroup B. Then distinct conjugates of B are disjoint and $\bigcup_{g \in G} B^g$ is generic in G. Furthermore, if L is a definable connected subgroup of G, then either L is in a conjugate of B or it intersects trivially every conjugate of B.*

PROOF. Let $g \in G \setminus N_G(B)$. If $B \cap B^g$ contains a nontrivial element x, then $C_G(x)^\circ$ is contained in a Borel subgoup K which has thus an infinite intersection with both B and B^g, contradicting the fact that B is isolated. So distinct conjugates of B are disjoint and as B is of finite index in its normalizer,

$$\mathrm{rk}\Big(\bigcup_{g \in G} B^g\Big) = \mathrm{rk}(B) + \mathrm{rk}(G/N_G(B)) = \mathrm{rk}(G).$$

Let now L be as in the second statement and suppose that $L \cap B^g$ is a nontrivial intersection for some $g \in G$. We claim that $L \cap B^g$ is normal in L. If it is not normal, then it must be finite as B is isolated. Pick then a nontrivial element x in this intersection; then $C_L(x)^\circ$ $(\leq L \cap N_G(B^g)^\circ = L \cap B^g)$ must be trivial, which proves the claim. Now, as L normalizes B^g, it is contained in B^g. ⊣

COROLLARY 4.2. *Let G be a simple FT-group which contains an isolated Borel subgroup B. Then every isolated Borel subgroup of G is a conjugate of B.*

PROOF. Let K be an isolated Borel subgroup of G. Then lemma 4.1 shows that $\bigcup_{g \in G} B^g$ and $\bigcup_{g \in G} K^g$ are both generic in G. As G is connected, we may assume, conjugating by an element of G if necessary, that B intersects K nontrivially, so that $B = K$ by lemma 4.1 again. ⊣

The problem of the conjugacy of Borel subgroups in a group of finite Morley rank remains open in general. But one can seriously wonder what happens in the preceding situation.

QUESTION 4.3. Let G be a simple FT-group which contains an isolated Borel subgroup B as above. Can one show that all Borel subgroups of G are conjugate to B?

§5. Proof of Theorem 1.2.

In this section we shall prove theorem 1.2. So let G be a simple FT-group in which every Borel subgroup is isolated. This assumption together with lemma 4.1 and corollary 4.2 leads to the same early analysis of bad groups of [15, 17], obtained in that case with the normalizer condition in nilpotent groups and a fusion argument. To summarize, if we

[5]The argument presented here is the most compact argument known by the author and is due to B. Poizat.

fix a Borel subgroup B, then distinct conjugates of B are disjoint, $\bigcup_{g\in G} B^g$ is generic in G and Borel subgroups are conjugate.

As a consequence of the preceding description, every element of G has an infinite centralizer, because if $g \in G$ had a finite centralizer, its conjugacy class would be generic and by connectedness a conjugate of g would be in B, thus its centralizer would be infinite. Furthermore every element g of G normalizes the unique conjugate of B which contains $C_G(g)^\circ$, so we have that $G = \bigcup_{g\in G} N_G(B)^g$.

After that, the main argument for bad groups (coming from [32]) consists in showing the absence of involutions (which allows then to show that Borel subgroups are selfnormalizing): a projective plane is constructed on the hypothetical set of involutions in such a way that three involutions are on the same line if and only if their product is an involution; this is a projective plane over an interpretable nonalgebraically closed field by a theorem of F. Bachmann and contradicts the fact that infinite fields of finite Morley rank are algebraically closed [30].

We could proceed in the same way with additional arguments needed for nonnilpotent Borel subgroups and a lemma[6] of O. Frécon which was crucial for the first version of the present paper. But we will give a shorter proof here, showing 'directly' that B is selfnormalizing by a reduction to bad groups (so that the deep argument with involutions becomes concealed). The tool for this reduction is the following fact extracted from [39]:

Fact 5.1. Let G be a solvable connected group of finite Morley rank. If G has a definable automorphism of prime order and with finitely many centralized elements, then G is nilpotent.

Proof. Let ϕ be the definable automorphism of prime order p. By [39], theorem 2.4.7, it suffices to show that the equality $g\phi(g)\dots\phi^{p-1}(g) = 1$ is generically satisfied in each definable characteristic subgroup of G. But if H is such a subgroup, then the application $h \longmapsto h\phi(h^{-1})$ from H to H has finite fibers, so the set of elements of its image, which satisfy the equality above, is generic in H. ⊣

So we now assume that $B < N_G(B)$ and we will get a contradiction.

Claim 5.2. *If x is an element of $N_G(B) \setminus B$ such that $C_B(x)$ is finite, then x is of finite order, the same as the order of its image in the finite quotient $N_G(B)/B$.*

Proof of claim. If n is the order of the image of x in $N_G(B)/B$, it suffices to show that $x^n = 1$. But if $x^n \neq 1$, then $C_G(x)^\circ$ ($\leq C_G(x^n)^\circ \leq B$) must be trivial, which is impossible. ⊣

[6]The lemma of O. Frécon says that if B is a solvable group of finite Morley rank and if x is an element of $B \setminus B^\circ$ such that $C_{B^\circ}(x)$ is finite, then $B^\circ = \{\, [b, x] : b \in B^\circ \,\}$ (in particular every element of the coset xB° is B°-conjugate to x).

Let now x be an element of $N_G(B) \setminus B$ such that the image of x in $N_G(B)/B$ is of prime order. If an element y of the coset xB had a finite centralizer in B, then B would be nilpotent by the claim and fact 5.1, so G would be a bad group and then B would be selfnormalizing. So $C_B(y)$ is infinite for each $y \in xB$, and $C_G(y)^\circ \leq B$. Consider now the definable set

$$X = \bigcup_{g \in G} (xB)^g.$$

If an element $y \in xB$ belongs to a conjugate B^g of B, then $B = B^g$ because they both contain $C_G(y)^\circ$ and $y \in B$, which is impossible. So $X \cap (\bigcup_{g \in G} B^g)$ is empty. Furthermore each $y \in X$ belongs to a single conjugate of xB: for if y were in xB and in $(xB)^g$ such that $B \neq B^g$, then as y is of prime order modulo B^g and has a finite centralizer in B^g (as $C_G(y)^\circ \leq B$), one could reduce the configuration to a bad group by the same argument as before. So, as each $y \in X$ is in a single conjugate of xB, one can see that X as well as $\bigcup_{g \in G} B^g$ are generic subsets, which is impossible in the connected group G as they are disjoint. This contradiction shows that B is selfnormalizing.

It follows then immediately that $B \cap B^g$ is nontrivial only if g is in B and that $G = \bigcup_{g \in G} B^g$. Thus $B < G$ is a full Frobenius group and theorem 1.2 is proved.

§6. Additional remarks. Obviously, a major difference between a 'classical' bad group with a nilpotent Frobenius complement (as defined and described in [15, 17]) and a 'generalized' bad group G with a solvable non-nilpotent Frobenius complement (as in the first case of theorem 1.2) is that the latter interprets an algebraically closed field K in its Borel subgroups by the Field Theorem. As this field K appears in definable sections of a solvable group without involutions, it is necessarily a bad field of characteristic different from 2 and where the infinite proper definable subgroup T of the multiplicative group $K^\times$ has no involution.

It has to be noticed that, in this case, G is interpretable in the field K (equipped with all the structure coming from G) by the results of E. Hrushovski [21, 35, corollaire 2.27]. And G is in fact definable in K by F. Wagner's result [38] of elimination of imaginaries of any field of finite Morley rank. One can wonder how to prove this definability here and notably if the involution of the multiplicative group of the algebraically closed field K can play any particular role in this configuration.

An argument of F. Delahan and A. Nesin [14, proposition 13.4] about involutory automorphisms of bad groups can be adapted word by word to the context of 'generalized' bad groups:

Proposition 6.1. *A simple FT-group such that $B < G$ is a full Frobenius group for some Borel subgroup B has no involutory nontrivial definable automorphism.*

Proof. Assume that i is such an automorphism. Then $C_G(i)^\circ$ is not trivial and $C_G(i)$ is in the Borel subgroup B of G containing $C_G(i)^\circ$. Furthermore $G = C_G(i)X$ where X is the set of elements inverted by i. One sees then easily that X is the union of $X \cap B$ together with Borel subgroups completely inverted by i and as $G = BX$, there exists such a Borel subgroup A completely inverted by i. There exists $x \in X$ such that $A = B^x$. Let then a be a nontrivial element of A such that $a \in C_G(i)^x$. Then $xax^{-1} = (xax^{-1})^i = x^{-1}a^{-1}x$, so x^{-2} inverts a, which gives an involution in G and thus a contradiction. ⊣

To conclude this paper, we wish to emphasize some general facts about groups without involutions. It is known from [7] that every such group is isomorphic to the automorphism group of a tournament, that is a set equipped with a complete and antisymetric binary relation. There exists a countable homogenous and universal tournament T which is the Fraïssé limit of the class of finite tournaments. (It is the natural analog for tournaments of the Rado graph.). Some properties of universality, for countable groups without involutions, of its automorphism group $\mathrm{Aut}(T)$ are shown in [29]. Notably every such group embeds into $\mathrm{Aut}(T)$ which is a simple group. Thus $\mathrm{Aut}(T)$ is a kind of monster group without involutions; it may be a place to build full Frobenius groups of finite Morley rank, or merely nonsolvable groups of finite Morley rank without involutions.

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UNIVERSITÉ CLAUDE BERNARD LYON-1
INSTITUT GIRARD DESARGUES
43 BLVD DU 11 NOVEMBRE 1918
69622 VILLEURBANNE CEDEX, FRANCE
E-mail: jaligot@desargues.univ-lyon1.fr
URL: http://www.desargues.univ-lyon1.fr/home/jaligot/jaligotF.html