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# SQUARE IN CORE MODELS

ERNEST SCHIMMERLING AND MARTIN ZEMAN

**Abstract.** We prove that in all Mitchell-Steel core models, $\square_\kappa$ holds for all $\kappa$. (See Theorem 2.) From this we obtain new consistency strength lower bounds for the failure of $\square_\kappa$ if $\kappa$ is either singular and countably closed, weakly compact, or measurable. (Corollaries 5, 8, and 9.) Jensen introduced a large cardinal property that we call *subcompactness*; it lies between superstrength and supercompactness in the large cardinal hierarchy. We prove that in all Jensen core models, $\square_\kappa$ holds iff $\kappa$ is not subcompact. (See Theorem 15; the *only if* direction is essentially due to Jensen.)

Jensen's square principle and its variants are ubiquitous in set theory.

DEFINITION 1 (Jensen). *Let $\kappa$ be an infinite cardinal. Then $\square_\kappa$ holds iff there exists a sequence $\langle C_\nu \mid \nu < \kappa^+\rangle$ such that if $\nu$ is a limit ordinal and $\kappa < \nu < \kappa^+$, then*

- *$C_\nu$ is a club subset of $\nu$,*
- *$\mathrm{type}(C_\nu) \leq \kappa$ and*
- *$C_\nu \cap \mu = C_\mu$ whenever $\mu \in \lim(C_\nu)$.*

Jensen also defined a global square principle called $\square$ (see Definition 19) that implies that $\square_\kappa$ holds for all $\kappa$. Jensen proved that $\square$ holds in $L$ and used $\square_\kappa$ in his construction of $\kappa^+$-Suslin trees, his proof of model theoretic transfer principles, and the list goes on. The fact that $\square$ holds in $L$ gained even more importance with Jensen's covering theorem for $L$. A frequently applied corollary is that if $\kappa$ is a singular cardinal and $\square_\kappa$ fails, then $0^\#$ exists.

If $0^\#$ exists, then $0^\#$ is not an element of $L$. Core models are generalizations of $L$ which may have more large cardinals. For the reasons mentioned above and many other well-known reasons, it is desirable to prove that $\square$ holds in as broad a class of core models as possible. Welch [27] showed that $\square$ holds in the Dodd-Jensen core model after Solovay had shown that the same holds in $L[U]$. Wylie [29] showed that $\square$ holds in the core model for measures of

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order 0. Schimmerling [16] showed that in the core models of Mitchell-Steel [13], $\square_\kappa$ holds whenever $o(\mu) < \mu^{++}$ for all $\mu \leq \kappa$. Schimmerling [16] also defined a hierarchy of weaker principles $\square_\kappa^\lambda$ (see Definition 16) and proved that some of these hold in the Mitchell-Steel core models. One such result was $\square_\kappa^2$ below $0^\P$, which was improved to $\square_\kappa$ by Jensen [5] and to $\square$ by Zeman [30]. Schimmerling [19] showed that $\square_\kappa^{<\omega}$ holds in the Mitchell-Steel core models. Building on these earlier results, the authors of this communication worked (first independently, then jointly) on the problem and contributed equally to its solution. (The theorems that we announce below include and go beyond those presented in the circulated notes Zeman [31] and in lectures by Schimmerling at Luminy in 1998.) Now, we have characterized those $\kappa$ for which $\square_\kappa$ holds in a given core model in terms of a large cardinal property of $\kappa$.

Space does not permit us to give the definition of *core model*,[1] but it is worth highlighting a few of the main points. Core models are proper class structures of the form

$$\langle L[\vec{E}], \in, \vec{E} \rangle$$

where $\vec{E} = \langle E_\alpha \mid \alpha \in \mathrm{OR} \rangle$ is a sequence of objects known as *extenders*. The *levels* of core models are *premice*, that is, structures of the form

$$\mathcal{J}_\alpha^{\vec{E}} = \langle J_\alpha^{\vec{E}}, \in, \vec{E} \restriction \alpha, E_\alpha \rangle.$$

One of the basic requirements on each extender $E_\alpha$ is that it codes a $\Sigma_1$ elementary embedding $\pi \colon \mathcal{J}_\alpha^{\vec{E}} \longrightarrow \mathcal{J}_\beta^{\vec{F}}$ where $\mathcal{J}_\beta^{\vec{F}}$ is also a premouse. The embedding is permitted to be the identity map; indeed, $L$ is a core model. Our convention here is that if the embedding determined by $E_\alpha$ is the identity map, then $E_\alpha = \emptyset$. The extender sequence of a premouse is also required to be *coherent* in the sense that if $\pi \colon \mathcal{J}_\alpha^{\vec{E}} \longrightarrow \mathcal{J}_\beta^{\vec{F}}$ is the embedding associated to $E_\alpha$, then $\vec{F} \restriction \alpha = \vec{E} \restriction \alpha$ and $F_\alpha = \emptyset$. Yet another requirement that is part of the definition is that if $\mu = \mathrm{crit}(\pi)$, then $\pi$ is completely determined by

$$\pi \restriction \left( J_\alpha^{\vec{E}} \cap \mathcal{P}(\mu) \right).$$

Equivalently, because of the strong form of $\mathsf{GCH}$ that holds in $L[\vec{E}]$, if

$$\lambda = (\mu^+)^{J_\alpha^{\vec{E}}},$$

[1]Historically, since Dodd and Jensen first introduced the expression *the core model*, it has been used to mean a certain canonical maximal inner model (also called $K$), which only exists under an anti-large cardinal hypothesis. When it exists, $K$ is a model of the form $L[\vec{E}]$. As the reader will note from the discussion that follows, we use the term *core model* to mean a model of the same general form as $K$, but not necessarily $K$ itself. This is not entirely standard.

then $\pi$ is determined by $\pi \upharpoonright \lambda$. Theoretically, such embeddings could witness that $\mu$ is superstrong or even slightly more than 1-extendible, but not $\lambda$-supercompact. However, it is not known how to construct a core model with a measurable Woodin cardinal, much less with a superstrong cardinal, so this is not a serious restriction today.

In the preceding paragraph, we were purposely vague about how $E_\alpha$ codes $\pi$ because there are at least two different ways that have been developed, one in Mitchell-Steel [13] and another, more recently, in Jensen [7]. (Jensen's approach was originally proposed by S. Friedman.) Our current opinion is that proofs of combinatorial principles flow more naturally for Jensen core models than Mitchell-Steel core models, or rather that analogous technicalities appear significantly later in the large cardinal hierarchy. However, the theory of $K$ below a Woodin cardinal has yet to be worked out for Jensen core models, and, therefore, it is important that we are able to prove our theorems for both kinds of core models. In either case, we can prove that $\square_\kappa$ holds in core models that are much more powerful than any we know how to construct from large cardinal hypotheses.

There is an important restriction that applies to the Mitchell-Steel core models but not to the Jensen core models. Let us say that an extender $E_\alpha$ is of *superstrong type* if

$$\pi(\mathrm{crit}(\pi)) < \alpha$$

where $\pi$ is the embedding of $\mathcal{J}^{\vec{E}}_\alpha$ coded by $E_\alpha$. In the Mitchell-Steel theory, it is assumed that no extender $E_\alpha$ is of superstrong type.

What turns premice of either flavor into *mice* is iterability in the sense of iteration trees.[2] The amount of iterability needed depends on the application. It is important for some of the corollaries that our results use only a modest form of iterability. For us, a premouse $\mathcal{J}^{\vec{E}}_\beta$ is *sufficiently iterable* iff whenever there is an elementary embedding from a countable premouse $\mathcal{P}$ into $\mathcal{J}^{\vec{E}}_\alpha$ for some $\alpha \leq \beta$, then $\mathcal{P}$ is $(\omega, \omega_1, \omega_1 + 1)$-iterable in the sense of Steel [26, 25]. This is just the amount of iterability needed to apply the methods of Neeman-Steel [15]. The only consequences of sufficient iterability that we use in our proofs are a few forms of condensation and solidity. For the purpose of this communication, we take *core model* to mean a proper class premouse all of whose levels have these condensation and solidity properties.

The following is one of our main results.

[2]Iteration trees on $V$ were first used by Martin and Steel; see [10]. There they prove, among other things, that if there are infinitely many Woodin cardinals, then Projective Determinacy (PD) holds. The theory of iteration trees is more general than that of linearly iterated ultrapowers; the rules for building iteration trees sometimes allow for an extender to be applied to an earlier model rather than to the model from which the extender comes. Iterability requires a strategy for choosing cofinal wellfounded branches at limit stages in the construction of an iteration tree.

THEOREM 2. *If $L[\vec{E}]$ is a Mitchell-Steel core model, then $L[\vec{E}]$ satisfies $\square_\kappa$ for all $\kappa$.*

We will list several immediate corollaries of Theorem 2. As the reader will see from these results, the extent to which we can apply Theorem 2 to obtain lower bounds on the large cardinal consistency strength of the failure of $\square_\kappa$ for certain kinds of $\kappa$ is limited by our ability to construct core models and prove covering theorems for them.

THEOREM 3 (Mitchell-Schimmerling [11]). *If the Steel core model $K$ exists, then $K$ correctly computes successors of singular cardinals.*[3]

Steel [26] showed that $K$ exists if, for example, every set has a sharp and there is no inner model with a Woodin cardinal. From this and Theorems 2 and 3, it follows logically that if every set has a sharp and $\kappa$ is a singular cardinal for which $\square_\kappa$ fails, then there is an inner model with a Woodin cardinal.[4] By limiting our attention to countably closed cardinals, we obtain a cleaner corollary.

DEFINITION 4. *A cardinal $\kappa$ is* countably closed *iff $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$.*

For example, $\aleph_\omega$ is countably closed iff $2^{\aleph_0} < \aleph_\omega$.

COROLLARY 5. *Let $\kappa$ be a singular countably closed cardinal. Suppose that $\square_\kappa$ fails. Then, for every $n < \omega$ and every $X \subseteq \kappa$, there is a transitive proper class $M_n(X)$ such that $X \in M_n(X)$ and $M_n(X)$ is a model of* ZFC *with $n$ Woodin cardinals. In addition, for every bounded $x \subset \kappa$, there is a transitive set $M$ such that $x \in M$ and $M$ is a model of* ZF - Replacement *with infinitely many Woodin cardinals.*

Schimmerling-Steel [20] proved the version of Corollary 5 for the weaker principle $\square_\kappa^{<\omega}$ (see Definition 16). Exactly the same methods work for $\square_\kappa$. What is important is that the witness to $\square_\kappa$ that Theorem 2 provides is *locally definable*, which is to say that $C_\nu$ is first-order definable over $\mathcal{J}_\beta^{\vec{E}}$ where $\beta$ is least such that $\nu$ is not a cardinal in $\mathcal{J}_{\beta+1}^{\vec{E}}$.

COROLLARY 6. *Let $\kappa$ be a singular countably closed cardinal. Suppose that $\square_\kappa$ fails. Then* PD *holds.*

Corollary 6 is immediate from Corollary 5 and the determinacy theorem of Martin-Steel [10]. To see this, let $X \subset 2^{\aleph_0}$ code a wellorder of $\mathbb{R}$. We will switch from the large cardinal hierarchy to the determinacy hierarchy to state some other consistency strength lower bounds.

THEOREM 7 (Schimmerling-Steel [20]). *If the Steel core model $K$ exists, then $K$ correctly computes successors of weakly compact cardinals.*

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[3]The proof of Theorem 3 is based on the corresponding earlier result of Mitchell-Schimmerling-Steel [12] for countably closed cardinals. Only the weaker result is used to prove Corollary 5.

[4]Since $\square_\kappa$ holds in $K$, if $K$ correctly computes $\kappa^+$, then $\square_\kappa$ holds in $V$.

It follows logically from various results mentioned above that if every set has a sharp and $\kappa$ is a weakly compact cardinal for which $\square_\kappa$ fails, then there is an inner model with a Woodin cardinal. Corollary 8 is a stronger result.

COROLLARY 8. *Suppose that $\kappa$ is a weakly compact cardinal and $\square_\kappa$ fails. Then every set of reals in $L(\mathbb{R})$ is determined.*

Schimmerling-Steel [20] proved the version of Corollary 8 for $\square_\kappa^{<\omega}$ and $\mathsf{PD}$. Woodin extended this to $\square_\kappa^{<\omega}$ and $\mathsf{AD}^{L(\mathbb{R})}$ using his Core Model Induction. The proof of Corollary 8 is no different. Like the proof of Corollary 5, that of Corollary 8 uses the local definability of the witness to $\square_\kappa$ which the proof of Theorem 2 provides.

COROLLARY 9. *Suppose that $\kappa$ is a measurable cardinal and $\square_\kappa$ fails. Then there is a transitive proper class model of* $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$.

The proof of Corollary 9 is an obvious combination of three results. The first is Woodin's theorem that $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ is consistent relative to the existence of a cardinal $\lambda$ such that 1) the Woodin cardinals are unbounded in $\lambda$ and 2) the $< \lambda$-strong cardinals are unbounded in $\lambda$. Thus, for the conclusion of Corollary 9, it suffices to prove that there is a core model that satisfies 1) and 2) for some $\lambda$. But otherwise, by Andretta-Neeman-Steel [1], if $\kappa$ is a measurable cardinal, then there is a core model that correctly computes $\kappa^+$. Their core model is essentially Steel's background certified core model, $K^c$. Corollary 9 would follow immediately from the two results just listed and Theorem 2, except for the fact that $K^c$ as constructed in Andretta-Neeman-Steel [1] is a Jensen core model rather than a Mitchell-Steel core model. To overcome this technicality, one simply applies Theorem 11 (below) instead of Theorem 2.[5]

COROLLARY 10. *Suppose that there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a transitive proper class model $M$ of* $\mathsf{ZFC}$ *such that $M$ has a Woodin cardinal that is a limit of Woodin cardinals and $M$ satisfies $\square_\kappa$ for all $\kappa$.*

Under the hypothesis of Corollary 10, Neeman [14] constructed a sufficiently iterable Mitchell-Steel core model with a Woodin limit of Woodin cardinals. Thus Corollary 10 is immediate from Neeman [14] and Theorem 2. Equiconsistency results closely related to Corollary 10 have been obtained by forcing. Burke [2] showed that $\square_\kappa$ holding for all $\kappa$ is consistent with the existence of a superstrong cardinal relative to the existence of a superstrong cardinal. Cummings [4] proved the analogous result for 1-extendible cardinals.

We now turn our attention to Jensen core models and a characterization of $\square_\kappa$ in these models in terms of a large cardinal property of $\kappa$. Let $L[\vec{E}]$ be a

[5] Woodin's property 1) + 2) is strictly within the large cardinal level handled by Andretta-Neeman-Steel [1], which they call *domestic*. So, in fact, if $\kappa$ is a measurable cardinal and $\square_\kappa$ fails, then there is a non-domestic premouse.

core model and $\kappa$ be a cardinal of $L[\vec{E}]$. Define $\mathcal{O}^{\vec{E}}(\kappa)$ to be the collection of $\alpha < (\kappa^+)^{L[\vec{E}]}$ such that $E_\alpha \neq \emptyset$ and, if $\pi$ is the embedding of $\mathcal{J}_\alpha^{\vec{E}}$ associated to $E_\alpha$, then $\pi(\text{crit}(E_\alpha)) = \kappa$. We remark that if $\kappa$ is a cardinal in $L[\vec{E}]$ and $\mathcal{O}^{\vec{E}}(\kappa) \neq \emptyset$, then there exists $\mu < \kappa$ such that $\mu$ is a superstrong cardinal in $L[\vec{E}]$.

THEOREM 11. *Suppose that $L[\vec{E}]$ is a Jensen core model that satisfies that $\mathcal{O}^{\vec{E}}(\kappa)$ is non-stationary in $\kappa^+$. Then $\square_\kappa$ holds in $L[\vec{E}]$.*

In comparing Theorems 2 and 11, one should not be misled into believing that a stronger form of $\square_\kappa$ holds in the Mitchell-Steel core models than in the Jensen core models. Let us remind the reader that in the Mitchell-Steel theory, no extender has superstrong type, while there is no corresponding smallness assumption on Jensen core models.

Jensen showed that our hypothesis in Theorem 11 is optimal.

PROPOSITION 12 (Jensen [8]). *Suppose that $L[\vec{E}]$ is a Jensen core model that satisfies that $\mathcal{O}^{\vec{E}}(\kappa)$ is stationary in $\kappa^+$. Then $\square_\kappa$ fails in $L[\vec{E}]$.*

Recall that Solovay [24] proved that $\square_\kappa$ fails if $\kappa$ is $\kappa^+$-supercompact, or even just $\kappa^+$-strongly compact. By a slight modification of Solovay's argument, Jensen proved Proposition 14 below; this was his first step towards the proof of Proposition 12.

DEFINITION 13 (Jensen [8]). *A cardinal $\kappa$ is* subcompact *iff for every $B \subseteq \kappa^+$, there exists $\mu < \kappa$, $A \subseteq \mu^+$ and an elementary embedding*

$$j\colon \langle H_{\mu^+}, \in, A\rangle \to \langle H_{\kappa^+}, \in, B\rangle$$

*with* $\text{crit}(j) = \mu$.[6,7]

PROPOSITION 14 (Jensen [8]). *If $\kappa$ is subcompact, then $\square_\kappa$ fails.*

PROOF. Deny. Let $B \subseteq \kappa^+$ code a $\square_\kappa$ sequence $\langle D_\alpha \mid \alpha < \kappa^+\rangle$. Select corresponding $\mu$, $A$ and $j$. Let $\langle C_\nu \mid \nu < \mu^+\rangle$ be the $\square_\mu$ sequence coded by $A$. Let $S = \{\nu < \mu^+ \mid \text{cf}(\nu) < \mu\}$. Then $j''S$ is $< \mu$-closed. Let $\beta = \sup(j''\mu^+)$. Then $\lim(D_\beta) \cap j''S$ is $< \mu$-closed and unbounded in $\beta$. Let $T = \{\nu < \mu^+ \mid j(\nu) \in \lim(D_\beta)\}$. Then $T$ is unbounded in $\mu^+$. Let $E = \bigcup\{C_\nu \mid \nu \in T\}$. Then $E$ is club in $\mu^+$ and $E \cap \nu = C_\nu$ for all $\nu \in T$. So $C_\nu$ has order type $> \mu$ for all sufficiently large $\nu \in T$, which contradicts the definition of $\square_\mu$. ⊣

[6]Note that $j(\mu) = \kappa$ follows.

[7]Jensen isolated the defining property of subcompactness but did not give it a name. In related work, he defined a cardinal $\kappa$ to be *quasicompact* iff for every $A \subseteq \kappa^+$, there exists $\lambda > \kappa$, $B \subseteq \lambda^+$ and an elementary embedding

$$j\colon \langle H_{\kappa^+}, \in, A\rangle \to \langle H_{\lambda^+}, \in, B\rangle$$

with $\text{crit}(j) = \kappa$.

THEOREM 15. *If $L[\vec{E}]$ is a Jensen core model, then $L[\vec{E}]$ satisfies*:

$$\square_\kappa \textit{ holds} \Longleftrightarrow \kappa \textit{ is not a subcompact cardinal.}$$

Let us describe the structure of the proof of Theorem 15. The ($\Rightarrow$) direction is an immediate consequence of Proposition 14. For ($\Leftarrow$), by Theorem 11, it is enough to show that, in $L[\vec{E}]$, if $\kappa$ is a cardinal and $\mathcal{O}^{\vec{E}}(\kappa)$ is stationary in $\kappa^+$, then $\kappa$ is subcompact. The rest of the argument is modeled on Jensen's proof of Proposition 12; it is short but, since it uses the fine structure of $L[\vec{E}]$, we do not include it here.

DEFINITION 16 (Schimmerling). *Let $\lambda \leq \kappa$ be cardinals. Then $\square_\kappa^{<\lambda}$ holds iff there exists a sequence $\langle \mathcal{C}_\nu \mid \nu < \kappa^+ \rangle$ such that if $\nu$ is a limit ordinal and $\kappa < \nu < \kappa^+$, then*

$$0 < |\mathcal{C}_\nu| < \lambda$$

*and, for every $D \in \mathcal{C}_\nu$,*

- *$D$ is a club subset of $\nu$,*
- $\mathrm{type}(D) \leq \kappa$ *and*
- *$D \cap \mu \in \mathcal{C}_\mu$ whenever $\mu \in \lim(D)$.*

By $\square_\kappa^\lambda$ we mean $\square_\kappa^{<\lambda^+}$. Note that $\square_\kappa^1$ is equivalent to $\square_\kappa$, and also that $\square_\kappa^\kappa$ is equivalent to Jensen's weak square principle $\square_\kappa^*$.

By Schimmerling [16], $\square_\kappa^*$ holds in all Mitchell-Steel core models; the same argument works for Jensen core models.

THEOREM 17. *Let $L[\vec{E}]$ be either a Mitchell-Steel core model or a Jensen core model. Then $L[\vec{E}]$ satisfies $\square_\kappa^*$ for all $\kappa$.*

Since Proposition 14 remains true if $\square_\kappa$ is replaced by $\square_\kappa^{<\kappa}$, we have the following equivalence.

THEOREM 18. *If $L[\vec{E}]$ is a Jensen core model, then $L[\vec{E}]$ satisfies*:

$$\square_\kappa \textit{ holds} \Longleftrightarrow \square_\kappa^{<\kappa} \textit{ holds.}$$

It is interesting to note that the dividing line between $\square_\kappa^*$ and $\square_\kappa^{<\kappa}$ is important elsewhere. By an elaborate forcing construction, it is shown in Cummings-Foreman-Magidor [3] that $\square_{\aleph_\omega}^*$ is consistent with a simultaneous reflection principle for stationary subsets of $\aleph_{\omega+1}$.[8] The same form of simultaneous reflection fails if $\aleph_\omega$ is a strong limit cardinal and $\square_{\aleph_\omega}^{<\aleph_\omega}$ holds; see Cummings-Schimmerling [4]. The consistency of other set theoretic principles with the square principles in this hierarchy has been examined; see the articles cited above for results, open problems and further references.

[8]Let $1 \leq n < \omega$ and $\langle S_i \mid i < \aleph_n \rangle$ be a sequence of stationary subsets of

$$\{\alpha < \aleph_{\omega+1} \mid \mathrm{cf}(\alpha) < \aleph_n\}.$$

Then there exists $\alpha < \aleph_{\omega+1}$ such that $\mathrm{cf}(\alpha) = \aleph_n$ and $S_i \cap \alpha$ is stationary in $\alpha$ for all $i < \aleph_n$.

Let SO be the class of singular ordinals and SC be the class of singular cardinals. With regard to the next definition, the only instances of $S$ that we will consider here are SO and SC.

Definition 19 (Jensen). *Let $S$ be a class of singular ordinals. Then $\Box^S$ holds iff there exists a class sequence $\langle C_\nu \mid \nu \in S\rangle$ such that if $\nu \in S$, then*

- *$C_\nu$ is a closed subset of $\nu \cap S$,*
- *if $\nu$ has uncountable cofinality, then $C_\nu$ is unbounded in $\nu$,*
- $\mathrm{type}(C_\nu) < \nu$ *and,*
- *if $\mu \in C_\nu$, then $C_\nu \cap \mu = C_\mu$.*

Jensen's global square principle $\Box$ is equivalent to $\Box^{\mathrm{SO}}$.

Theorem 20 (Jensen).

$$\Box \iff \left(\Box^{\mathrm{SC}} \wedge \forall\kappa\Box_\kappa\right)$$

Theorem 21 (Zeman). *If $L[\vec{E}]$ is a Jensen core model, then $\Box^{\mathrm{SC}}$ holds in $L[\vec{E}]$.*

We have said little about the proofs of Theorems 2 and 11 and what makes them different from the proof of $\Box_\kappa$ in $L$. Assume that $V = L[\vec{E}]$ and $\kappa$ is a cardinal. The main difficulty is typified by the case in which $C_\alpha$ is being defined, $\kappa$ is the largest cardinal in $\mathcal{J}^{\vec{E}}_\alpha$ and $E_\alpha$ is an extender with $\mathrm{crit}(E_\alpha) < \kappa$. Since $\kappa$ is a cardinal, $E_\alpha$ is a total extender over $L[\vec{E}]$. If we had given the entire definition of the term *premouse*, then the reader would see that there is a club subset of $\alpha$ of order type $\mathrm{crit}(E_\alpha)^+$ that is $\Sigma_1$ definable over $\mathcal{J}^{\vec{E}}_\alpha$. The idea of adapting Jensen's proof that $\Box_\kappa$ holds in $L$ does not seem to work because the kind of condensation that Jensen uses in his argument is easily seen to fail for $\mathcal{J}^{\vec{E}}_\alpha$. Instead, we proceed as follows. Say $E_\alpha$ corresponds to the embedding $\pi\colon \mathcal{J}^{\vec{E}}_\alpha \longrightarrow \mathcal{J}^{\vec{F}}_\beta$. There may be as many as $\kappa$ different ways to factor $\pi$ as

$$\begin{tikzcd}
\mathcal{J}^{\vec{E}}_\alpha \arrow[r, "\pi"] \arrow[dr] & \mathcal{J}^{\vec{F}}_\beta \\
& \mathcal{J}^{\vec{E}}_\gamma \arrow[u, "\sigma"']
\end{tikzcd}$$

where $\mathcal{J}^{\vec{E}}_\gamma$ has cardinality $< \kappa$ and $\mathrm{crit}(\pi) < \mathrm{crit}(\sigma) < \kappa$. Given such a diagram, let $G$ be the extender derived from $\sigma$. We remark that $G$ is a partial extender in the sense that $G$ does not act on all of $L[\vec{E}]$. It turns out that, from $G$, there is a natural way to define a club subset of $\alpha$ of order type $(\mathrm{crit}(G)^+)^{\mathrm{J}^{\vec{E}}_\gamma}$. The proofs of various instances of $\Box^\lambda_\kappa$ in Schimmerling [16] and [19] amount to bounding by $\lambda$ the number of factor embeddings that must be considered and letting $\mathcal{C}_\alpha$ be the collection of club subsets of $\alpha$ associated to $E_\alpha$ and the various $G$'s. In the proofs of Theorems 2 and 11, we choose either $E_\alpha$ or one of the $G$'s and show that our choice is canonical. The details can be found in Schimmerling-Zeman [22]. We are optimistic that the full combinatorial structure of $L$ can be generalized to $L[\vec{E}]$ by our new methods.

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DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PA 15213-3890, USA
*E-mail*: eschimme@andrew.cmu.edu

INSTITUT FÜR FORMALE LOGIK
WÄHRINGER STRAßE 25
A-1090 WIEN, AUSTRIA
and
MATHEMATICAL INSTITUTE SAV
ŠTEFÁNIKOVA 49
814 73 BRATISLAVA, SLOVAKIA
*E-mail*: zeman@logic.univie.ac.at