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REVIEWS

The Association for Symbolic Logic reviews selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Geoffrey Hellman, Thomas Jech, Wolfram Pohlers, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VI 94" refers to the review beginning on page 94 in volume 6 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 192***2*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218). Similar references containing the fraction $\frac{1}{2}$ or a decimal point (such as $70\frac{1}{2}$***1*** or 382***7.1***) are to *Additions and corrections to A bibliography of symbolic logic* (the JOURNAL, vol. 3, pp. 178–212).

In the light of logic, by Solomon Feferman, Logic and computation in philosophy, Oxford University Press, New York, Oxford, etc., 1998, xii + 340 pp.—therein:

SOLOMON FEFERMAN. *Deciding the undecidable: wrestling with Hilbert's problems*. Pp. 3–27.

SOLOMON FEFERMAN. *Infinity in mathematics: Is Cantor necessary?* Pp. 28–73. (Reprinted with minor changes from ***L'infinito nella scienza*** (***Infinity in science***), edited by Giuliano Toraldo di Francia, Istituto della Enciclopedia Italiana, Rome 1987, pp. 151–190.)

SOLOMON FEFERMAN. *The logic of mathematical discovery versus the logical structure of mathematics*. Pp. 77–93. (Reprinted with minor changes from ***PSA 1978***, Volume 2, ***Symposia***, edited by Peter D. Asquith and Ian Hacking, Philosophy of Science Association, East Lansing 1981, pp. 309–327.)

SOLOMON FEFERMAN. *Foundational ways*. A reprint with minor changes of JSL LVII 1478(1). Pp. 94–104.

SOLOMON FEFERMAN. *Working foundations—'91*. A revised version of JSL LVII 1478(2). Pp. 105–124. (Reprinted with minor changes from ***Bridging the gap, Philosophy, mathematics, and physics, Lectures on the foundations of science***, edited by Giovanna Corsi, Maria Luisa Dalla Chiara, and Gian Carlo Ghirardi, Boston studies in the philosophy of science, vol. 140, Kluwer Academic Publishers, Dordrecht, Boston, and London, 1993, pp. 99–124.)

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Solomon Feferman. *Gödel's life and work*. A partial reprint of JSL LV 340. Pp. 127–149.

Solomon Feferman. *Kurt Gödel: conviction and caution*. A reprint with minor changes of JSL LVIII 365. Pp. 150–164.

Solomon Feferman. *Introductory note to Gödel's 1933 lecture*. Pp. 165–173. (Reprinted with minor changes from ***Collected works***, Volume III, ***Unpublished essays and lectures***, by Kurt Gödel, edited by Solomon Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons, and Robert M. Solovay, Oxford University Press, New York and Oxford 1995, pp. 36–44.)

Solomon Feferman. *What does logic have to tell us about mathematical proofs?* Pp. 177–186. (Reprinted with minor changes from ***The mathematical intelligencer***, vol. 2 no. 1 (1979), pp. 20–24.)

Solomon Feferman. *What rests on what? The proof-theoretic analysis of mathematics*. Pp. 187–208. (Reprinted with minor changes from ***Philosophie der Mathematik, Akten des 15. internationalen Wittgenstein-Symposiums***, Volume I, edited by Johannes Czermak, Schriftenreihe der Wittgensteingesellschaft, vol. 20/I, Verlag Hölder-Pichler-Tempsky, Vienna 1993, pp. 147–171.)

Solomon Feferman. *Gödel's Dialectica interpretation and its two-way stretch*. Pp. 209–225. (Reprinted with minor changes from ***Computational logic and proof theory, Third Kurt Gödel Colloquium, KGC '93, Brno, Czech Republic, August 24–27, 1993***, edited by Georg Gottlob, Alexander Leitsch, and Daniele Mundici, Lecture notes in computer science, vol. 713, Springer Verlag, Berlin, Heidelberg, and New York, 1993, pp. 23–40.)

Solomon Feferman. *Infinity in mathematics: Is Cantor necessary? (Conclusion)*. Pp. 229–248. (Reprinted with minor changes from ***L'infinito nella scienza*** (***Infinity in science***), edited by Giuliano Toraldo di Francia, Istituto della Enciclopedia Italiana, Rome 1987, pp. 190–209; also in ***Philosophical topics***, vol. 17 no. 2 (1989), pp. 23–45).

Solomon Feferman. *Weyl vindicated: Das Kontinuum seventy years later*. A reprint of JSL LVIII 1085 with minor changes and added postscript. Pp. 249–283.

Solomon Feferman. *Why a little bit goes a long way: logistical foundations of scientifically applicable mathematics*. Pp. 284–298. (Reprinted with minor changes from ***PSA 1992***, Volume 2, ***Symposia and invited papers***, edited by David Hull, Micky Forbes, and Kathleen Okruhlik, Philosophy of Science Association, East Lansing 1993, pp. 442–455.)

In this book, ***In the light of logic***, Feferman collects a number of essays and other articles of historical and philosophical nature, which had previously been scattered in a number of different and (at least in a few cases) not widely accessible publications. This is not—properly speaking—a logic book, but it is a book in which logic occurs at every step to cast some light on some subtle distinction or to bring weight to a philosophical or conceptual point.

While some parts of the book are quite technical, the arguments are framed in such a way as to be accessible to non-specialists, and many parts are indeed adequate for a general audience. Accordingly, the book does not offer complete presentations of any of Feferman's numerous technical contributions. For those, the interested reader will have to follow the references to academic journals and scholarly publications given in the book's extensive bibliography.

Rather, with this book Feferman intends to offer a comprehensive view and a spirited defense of his *predicativist* position in the philosophy of mathematics. Predicativism is the view that one should only allow entities that are definable without making any references to totalities of which they themselves are members. Definitions that violate this requirement are called impredicative: for example, the definition of the maximum of a bounded set of reals is impredicative because it involves quantification over the collection of the upper bounds of the given set. Of course, if we assume that there is a totality of mathematical entities given *sub specie aeternitatis*, then impredicative definitions turn out to be perfectly justifiable:

after all, from this point of view, each mathematical statement has a perfectly determined truth-value, so that definitions involving unbounded quantification over arbitrary entities are hardly a problem. It follows that predicativism goes hand-in-hand with a rejection of the unbridled power-set principle (the idea that whenever a set A is given, so is the collection of all subsets of A), as well as a dynamical conception of mathematical reality, according to which mathematical entities are not given once and for all, but can be conceived as the outcome of generative processes.

It is worth noting that, contrary to the grand foundational schemes of the 1920's and 1930's (logicism, intuitionism, formalism), predicativism is not a single philosophical position, but rather a family of positions, each relative to a given conceptual framework taken as basic. Predicativists allow only entities that are non-circularly definable, *given the entities of the basic framework*. For instance, a well-known version of predicativism, first proposed by Hermann Weyl (one of the heroes of Feferman's book), regards unbounded quantification over the natural numbers as legitimate, but not quantification over *sets* of natural numbers. As a result, only sets of natural numbers that are arithmetically definable are allowed. One might think that this is a rather Draconian restriction, but perhaps the main point of the articles under review is that most if not all scientifically applicable mathematics can be formalized in a predicatively justified framework inspired by Weyl's approach.

The book comprises fourteen chapters, divided into five parts: (I) *Foundational problems*; (II) *Foundational ways*; (III) *Gödel*; (IV) *Proof theory*; and (V) *Countably reducible mathematics*. In addition, the book contains a list of defined symbols, an extensive bibliography, and an index mostly of names. Since the various chapters were originally published separately, there is a certain amount of overlap. For instance, Chapters 4 and 5 (*Foundational ways* and *Working foundations*) are a slimmed down and, respectively, an expanded version of an article originally published in the journal ***Synthese***. The book is quite pleasantly written: Feferman's prose is clear and incisive, never redundant; altogether this makes the book a great read. It is worth mentioning that in over 300 pages, there appear to be at most a handful of typographical errors, the most bewildering of which are a couple of remnants of un-escaped LaTeX code that found their way into the printed version (pp. 36 and 252).

As already mentioned, the book contains a powerful argument in favor of a predicativistic conception of mathematics; the argument is complemented and supported by a number of expository or historical articles, such as the ones found in Parts I and III. Similarly, Chapters 11 (*Gödel's Dialectica interpretation and its two-way stretch*) and 13 (*Weyl vindicated: Das Kontinuum seventy years later*) play a somewhat auxiliary role with respect to the main thrust of the book. It seems useful, therefore, to touch upon these parts first, in order later to present an unobstructed account of the main argument.

Part I, *Foundational problems*, presents two expository pieces, the second of which, *Infinity in mathematics: Is Cantor necessary?* is a particularly accessible presentation of a number of foundational issues. After giving a brief account of the origins and early development of set theory with Cantor and Zermelo, Feferman points out questions and problems connected to the underlying Platonism, and presents the “solutions” of Brouwer, Hilbert, and Weyl. While Brouwer's proposal is perceived as too restrictive, and Hilbert's as ultimately failing due to Gödel's incompleteness theorems, Feferman's sympathies go to Weyl's attempt to give a predicatively sound account of modern mathematics which is ultimately reducible to quantification over the natural numbers. The chapter's breadth and accessibility, together with Feferman's remarkably clear prose, make it especially suitable as introductory reading in a philosophy of mathematics course.

Part III, *Gödel*, offers an interesting account of Gödel's life and work, emphasizing his philosophical views. As editor-in-chief of Gödel's ***Collected works***, Feferman is in a sense in a unique position to draw upon extensive material from Gödel's *Nachlaß*. Two points are of particular philosophical interest here. In *Kurt Gödel: conviction and caution*, Feferman points out Gödel's reluctance, while in Vienna, to make explicit his realist views in the philosophy of mathematics (the Vienna environment, dominated by the verificationist philosophy of the *Wiener Kreis*, in turn inspired by Hilbert's formalist program, was probably a factor). This reluctance is likely behind the fact that, although Gödel relied on non-constructive notions such as arithmetical truth when deriving the incompleteness theorems, he took pains to purge any reference to truth from his published work. What is even more surprising perhaps, is that, as Feferman points out, Gödel had discovered the undefinability of arithmetical truth already in 1931, at least a couple of years before Tarski's famous theorem to the same effect.

The other unexpected bit of history contained in Feferman's account is a passage in Gödel's invited address to a meeting of the Mathematical Association of America, held in Cambridge in 1933. Of course, Gödel's long-held Platonist sympathies are well known, and although as we have seen he might have been reluctant in publicizing them while in Vienna, he unabashedly claimed to have held them all along after coming to the United States in the late 1930's (a time when, coincidentally, the Vienna Circle was undergoing its own diaspora and the fortunes of Hilbert's original program had all but faded). All the more surprising, then, is the passage from the 1933 lecture: "... our axioms [i.e, the axioms of Zermelo–Fraenkel set theory], if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent." Based on the evidence available from Gödel's publications and the *Nachlaß*, this passage can be neither explained nor explained away. The historian's only option is to take it at "face value," and attribute it to a momentary period of doubt; this is where, with Feferman, we leave the matter.

Completing the expository part (although clear-cut distinctions are hard to draw), are the two chapters on Gödel's ***Dialectica*** interpretation (JSL XXV 351) and Weyl's ***Das Kontinuum*** (192**2**). These two pieces are exemplary in their level of detail both for their clarity and historical rigor; insofar as they fall within the history of mathematical ideas, these pieces contribute to set a very high standard indeed, and especially the Weyl article is a small classic in its own. The two articles are similar in structure: both move from a historical point—an analysis and reconstruction of Gödel's and Weyl's original proposals—and both follow the contemporary developments that those proposals originated. Both contribute to Feferman's main point, that results, methods, and tools originating from the finitist, constructivist, or predicativist tradition represent a viable alternative to Platonistically inspired mathematics, especially set theory.

We now come to the main philosophical argument of the book, developed in Parts II, IV, and V. In Part II, *Foundational ways*, Feferman presents his view of the nature of the foundations of mathematics. After an appraisal of Lakatos's conception of mathematical practice and a comparison to Pólya's more detailed, but less sweeping account (Chapter 3), Feferman proceeds to present his own view in the two already mentioned articles *Foundational ways* and *Working foundations* (Chapters 4 and 5). Shunning grand proposals purporting to provide foundations for a field that, in the view of most working mathematicians, does not need any, Feferman identifies a number of foundational activities that, while less general, provide an arena for logic, philosophy, and mathematics to interact in fruitful ways. In this sense, rather than *philosophy of mathematics*, these activities can be regarded as a form of Russell's *mathematical philosophy*, i.e., that kind of philosophical reflection that is *continuous with* mathematics rather than taking mathematics as object of external investigation.

It is worth spending some time identifying some of these foundational ways. The first is *conceptual clarification*. This takes places once a subject has reached a certain level of maturity: examples from different fields are the concept of continuous function, Tarski's definitions of truth and satisfaction, the Turing–Post notion of computable function. To the philosophical eye these look like as many instances of Carnap's idea of *explication*, i.e., the sharpening of a pre-existing but imprecise concept. In this respect, Feferman notices that an informal concept still in need of an explication is the one of *identity of proofs*: in spite of there being a well-developed proof theory (to which Feferman himself has made fundamental contributions), there is no precise account of what it means to say that two proofs are essentially different or the same.

A second foundational way is the analysis of supposedly problematic concepts and principles through an *interpretation*. For instance, one provides an interpretation of complex numbers as pairs of reals or of the axiom of choice in the constructible universe. Alternatively, instead of providing interpretations for problematic concepts, one can provide *substitutes* for them: this is the case for instance of the usual ε-δ definition of limit, which dispenses with the problematic notion of *infinitesimal*, or the Bernays–Gödel theory of sets and classes, which allows the replacement of the notion of a category of all categories with the "large" category of all "small" categories.

Axiomatization is perhaps the foundational activity *par excellence*. Coming late in the development of a subject, systematization and axiomatization can sometimes be little more than a bookkeeping exercise. Beside the classical characterizations of the basic number systems, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, the axiomatization of several different constructive theories is also worth mentioning.

But the most innovative foundational activity, in Feferman's view, is the kind of generalization he refers to as *reflective expansion*. The generalization from the Euclidean space $\mathbb{R}^3$ to arbitrary n-dimensional spaces $\mathbb{R}^n$ can be viewed as a reflective expansion, but the most interesting applications of the idea concern the progression from one given body S of concepts and principles to a further body S', whose concepts and principles are regarded as "implicit" in those of S. In later work, Feferman will refer to S' as the *unfolding* of S (see for instance Feferman's *Gödel's program for new axioms: Why, where, how, and what?* in ***Gödel '96***, Lecture notes in logic, no. 6, Springer 1996, pp. 3–22). The unfolding process can be viewed as the attempt to answer the vaguely normative question, first formulated by Kreisel: What concepts and principles ought one to accept if one has accepted given concepts and principles?

An important example of unfolding is the case of *predicativity*, whose analysis is one of Feferman's main contributions. Suppose we accept—with Weyl—the natural number system $\mathbb{N}$ as a completed totality, therefore allowing unrestricted quantification over $\mathbb{N}$ as meaningful. Then, any property of natural numbers expressible by a first-order formula of arbitrary quantifier complexity is well defined, and thus determines a subset of $\mathbb{N}$. In turn, the collection of subsets one allows determines the strength of the corresponding induction principle. In this manner, one can proceed through a number of stages, where at each stage one allows properties expressible by quantifying over previously determined collections of properties. As independently discovered by Feferman and Schütte, this process gives out at the ordinal Γ_0, which has since come to be regarded as the ordinal characterizing predicativity.

This idea, together with the variations proposed by Feferman and his collaborators over the years, embodies an alternative view of the foundations of mathematics. Here the contrast is not only to foundational programs of the early 1900's—logicism, formalism, and intuitionism as originally formulated have long been abandoned. The contrast, at least implicitly, is also to the dominating trend in logic and foundations to reduce every aspect of mathematical practice to the omni-comprehensive framework of Zermelo–Fraenkel set theory and its extensions by various large cardinal axioms. As Feferman points out, there is room for a point of view that emphasizes ontological frugality over extravagance, and that conservatively tries to identify the resources that are required for carrying out given tasks, with a particular attention to the needs of science.

This ideal is substantiated in the next two parts of the book, *Proof theory* (IV) and *Countably reducible mathematics* (V). In the article *What does logic have to tell us about mathematical proofs?* Feferman introduces an analogy that is suggestive of his conception of the role of logic in foundations. A logical theory is comparable to a physical theory, in that logic deals with the structure of mathematics pretty much in the same way in which physics deals with the structure of physical reality. The tools employed are formal systems in one case, and differential equations in the other. Accordingly, logic provides a structural analysis of mathematics that is independent of its traditional subdivision into geometry, algebra, analysis, number theory, etc. Orthogonally to these subdivisions, logic tries to characterize the activity of idealized mathematicians as they are engaged in the task of drawing conclusions from given assumptions. In this respect, the tools, ideas, methods, and insights gained from proof theory—arguably the most "philosophical" of all the branches of symbolic logic—play a crucial role.

As argued in *What rests on what? The proof-theoretic analysis of mathematics*, the theory of proofs provides us with a fine-grained metric for the dependency relations among various parts of formalized mathematical theories, determining exactly in what sense one is "reducible" to another. Indeed, there are at least two different notions of reduction that logic affords us. One is the already mentioned *interpretability* of one theory into a stronger one. The other is a notion of *proof-theoretical* (or *foundational*) reduction. This kind of reduction moves in the opposite direction from the other one: a stronger theory is reduced to a weaker one by establishing certain conservativity results. Feferman refers to the task of establishing foundational reductions of this kind as a *relativized Hilbert program*.

The basic idea of a proof-theoretical reduction of a theory T_1 to a theory T_2, for a class Φ of formulas, is that every proof of a formula $\varphi \in \Phi$ in T_1 can be computably translated into a proof of φ in T_2, and moreover this very fact can be represented in the weaker theory T_2 itself. Perhaps the first example of such a reduction, due to Ackermann, is the reducibility of first-order arithmetic with the quantifier-free induction schema to primitive recursive arithmetic. This early result has been strengthened many times in many different directions; particularly interesting is the case in which a second-order theory is reduced to a first-order one. The easiest and most representative result of this kind is the fact that arithmetic with a comprehension principle for arithmetically definable sets and the second-order induction axiom is reducible to the first-order theory of Peano arithmetic.

The conceptual import of these results is that fairly strong arithmetical systems including various second-order comprehension principles and various forms of induction can be *justified* on the basis of finitistically (in the case of primitive recursive arithmetic) or, in any case, predicativistically (in the case of Peano arithmetic) acceptable frameworks.

This fact can be used to carry out a critical assessment of the the so-called Gödel-doctrine: the idea, first formulated by Gödel in a footnote to his incompleteness paper, that in order to decide more and more arithmetical propositions, one has to ascend the (cumulative) type hierarchy well into the transfinite. The Gödel-doctrine is often invoked as an argument in favor of embracing the "Cantorian transfinite." In particular, a number of results beginning with the famous Paris–Harrington theorem, and culminating with Harvey Friedman's recent work, have purported to show that the "higher infinite" is necessary for ordinary mathematical practice. The main idea here is that a finite combinatorial statement is produced which is (a) of independent interest for the "working mathematician," and (b) not settled by some "standard" theory in which mathematical practice is formalized. In the case of the Paris–Harrington theorem, the statement in question is a finite version of Ramsey's theorem, and it is shown to be independent of Peano arithmetic. But Friedman's recent work, especially on Boolean relation theory (BRT), has pushed this much further, by producing statements that appear more and more "natural" from the point of view of the working mathematician ("natural" in the same sense in which Gödel's undecidable sentence might appear contrived and ad hoc), and whose proofs require stronger and stronger principles well beyond the reach of Zermelo–Fraenkel set theory, in particular, large cardinal hypotheses. (The reader interested in keeping up with Friedman's ongoing research in this direction could usefully consult the archive of the "Foundations of mathematics" (FOM) discussion list, at http://www.math.psu.edu/simpson/fom.)

Feferman has two rejoinders to these claims. First, referring among other things to his own work, Feferman shows that (in at least some initial cases of the Paris–Harrington type) ascent of the type hierarchy (objectionable in his own view because it presupposes a well-defined notion of power-set) can be replaced by ascent of another, less ontologically committed kind. This alternative ascent is achieved by adjoining to a given theory either a truth-predicate or appropriate reflection principles. In both cases, one succeeds in going beyond Peano arithmetic without committing to an intrinsically uncountable framework. More importantly, Feferman's second rejoinder questions Friedman's claim that the combinatorial principles are natural and readily understandable—and that therefore the intrinsic necessity of strong systems for everyday finitary mathematics has been established. On closer scrutiny one sees that Friedman's results (important and difficult as they are) establish the equivalence of the combinatorial statements not with the large cardinal hypotheses themselves, but with their 1-consistency, i.e., the claim that all purely existential number-theoretical statements provable from the hypotheses are true. The leap from 1-consistency to truth is perhaps not insurmountable, but only if one is already committed to there being a fact of the matter as to the truth of the large cardinal hypotheses. Moreover, contrary to the cases of the Paris–Harrington type, the combinatorial statements themselves cannot be established by ordinary mathematical reasoning, but only under the assumption of the 1-consistency of the large cardinal hypotheses. To turn around and claim that the combinatorial statements provide evidence for those hypotheses is, for Feferman, an evident *petitio principii*.

The Feferman–Friedman debate is perhaps one of the most interesting recent exchanges in the philosophy and foundations of mathematics. The embers of the discussion, although smoldering for some time, have been recently rekindled with the publication of Feferman's expository paper *Does mathematics need new axioms?* (***The American mathematical monthly***, vol. 106 (1999), pp. 99–111) and again at the 2000 meeting of the Association for Symbolic Logic in Urbana, Illinois, where Feferman debated the issues with Harvey Friedman, Penelope Maddy, and John Steel (this BULLETIN, vol. 6 (2000), pp. 401–446).

Returning to the book's main argument, a potential challenge to Feferman's outlook on the foundations of mathematics is provided by the so-called Quine–Putnam *indispensability arguments* for mathematical realism. The idea is that such realism is justified because the mathematical entities whose existence is asserted—including strongly inaccessible stages in the cumulative hierarchy or various kinds of large cardinals—are necessary for the development of modern science. Such arguments have been variously challenged, perhaps most recently by Maddy. Feferman is not here concerned with undermining the arguments' validity (although he does share many of Maddy's reservations). Rather, Feferman focuses on the fact that the proponents of these arguments seldom consider exactly how much mathematics is needed for science. Once this assessment is carried out—not a light task, and one that has engaged a number of researchers, including Feferman, for a long time—it will become apparent that "surprisingly meager (in the proof-theoretical sense) predicatively justified systems suffice for the formalization of almost all, if not all, scientifically applicable mathematics" (p. 285). The system in question is Feferman's system W of variable types (with a non-constructive least-number search operator), itself inspired by Weyl's attempt to develop a system of "arithmetical analysis." (There are, indeed, bits of mathematics that cannot be represented in W, e.g. those that refer in an essential way to non-measurable objects or non-separable Hilbert spaces. But their use in contemporary physics is controversial, and they might in fact not be needed at all.) The system W is proof-theoretically reducible to Peano arithmetic, of which it is a conservative extension. It follows that "acceptance of W ... does not commit one to a platonistic ontology ... , though the platonist is free to understand W in those terms" (p. 296).

The argument is now complete. After having identified a range of appropriate foundational activities, and in particular the notion of the *unfolding* of a given conceptual framework, Feferman proceeds to point out the role of proof-theoretical reductions. In turn, proof-theoretical reduction is applied to justify, in predicativistically acceptable terms, a particular formal system W that is both flexible and powerful enough to represent contemporary scientific mathematics.

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JOSÉ FERREIRÓS. ***Labyrinth of thought. A history of set theory and its role in modern mathematics.*** Science networks, vol. 23. Birkhäuser Verlag, Basel, Boston, and Berlin, 1999, xxi + 440 pp.

The "labyrinth" of the title is from Jorge Luis Borges's ***La cifra*** (Madrid 1981): "I descried, through the pages of Russell, the doctrine of sets, the *Mengenlehre*, which postulates and explores the vast numbers that an immortal man would not reach even if he exhausted his eternities counting, and whose imaginary dynasties have the letters of the Hebrew alphabet as ciphers. It was not given to me to enter that delicate labyrinth" (p. xi). Notwithstanding and braving the daunting complexities of this labyrinth, José Ferreirós has written a magisterial account of the history of set theory which is panoramic, balanced, and engaging. Not only does this book synthesize much previous work and provide fresh insights and points of view, but it also features a major innovation, a full-fledged treatment of the emergence of the set-theoretic approach in mathematics from the early nineteenth century. This takes up Part One of the book. Part Two analyzes the crucial developments in the last quarter of the nineteenth century, above all the work of Cantor, but also of Dedekind and the interaction between the two. Lastly, Part Three details the development of set theory up to 1950, taking account of foundational questions and the emergence of the modern axiomatization. The author's English is flowing and lucid, though from time to time one becomes aware of a foreign sensibility.

After an informative chapter on German mathematics in the early nineteenth century that emphasizes emergent foundational viewpoints and a positive attitude toward infinity, Ferreirós in Part One focuses on a new fundamental notion, Riemann's manifolds (*Mannigfaltigkeit*). Riemann in his *Habilitationsvortrag* of 1854 famously provided the foundations of differential geometry and liberalized notions of physical space. Ferreirós deals at length with the *Vortrag* and its antecedents, especially how the notion of manifold drew on the traditional logical notion of extension, or class. The point is made that the *Habilitationsvortrag* became generally known only after its posthumous publication by Dedekind in 1868, together with the famous paper on trigonometric series that included the definition of the (Riemann) integral. Cantor's seminal work occurred just a few years later, and it cannot be a happenstance that his early term for *set* is Riemann's *Mannigfaltigkeit*. The other main focus of Part One is Dedekind's set-theoretic approach to algebra and the emergence of algebraic number theory. Soon after his own *Habilitationsvortrag* of 1854, Dedekind was making fundamental moves toward abstract algebra based on sets and structure-preserving mappings, and by the early 1870's he was developing his theory of ideals, avowedly infinite collections. Ferreirós concludes Part One with an astute account of the formulations of the real numbers and the development of topological and measure-theoretic ideas in analysis, the focus being on the 1870's and the increasing elaboration of the set-theoretic approach.

In his Part Two, *Entering the labyrinth*, Ferreirós accords a central place to Cantor for his investigations of the transfinite, but also places Dedekind prominently on the stage. Cantor's revolutionary work on cardinality is given a deft and vivid presentation that interestingly weaves in his episodic correspondence with Dedekind, a correspondence that Ferreirós has specifically worked over in detail. Next Dedekind's formulation of the natural numbers is given an autonomous treatment, viewed as precipitating sets and mappings as a foundation for mathematics. Emphasizing how this approach is logical, Ferreirós writes significantly to Dedekind's "logicist" program, rather going against historiographic usage. Next, Cantor's main work on the continuum hypothesis through his mature development of ordinal numbers is adroitly presented. While Cantor's well-known political difficulties are described, it is argued that the reception of Cantor's work was not as hostile as sometimes depicted nor was its psychological toll on Cantor as heavy.

In Part Three, Ferreirós first discusses the period of "Diffusion, crisis, and bifurcation" of set theory, 1890–1914. He regards the "crisis" at the turn of the century not so much as consternation caused by the paradoxes for any presumed foundations for mathematics, but rather as a debate over the acceptability of the new abstract mathematics, with the axiom of choice a specific focus. The early work of Zermelo on the axiom and axiomatization generally figures centrally in Ferreirós's account, but so does Russell's theory of types as a consistent rendition of the principle of comprehension, thus according Russell's theory a fully complementary role in a "bifurcation" Ferreirós sees in the development of set theory during this period. Ferreirós provides a synthetic account of the Zermelo vs. Russell approaches in terms of changing conceptions of what logic is and of emerging technical simplifications like the simple theory of types. How can a reconciliation be effected? This would be through the "radical proposal" of Weyl and Skolem to use first-order logic as a natural underlying logic for mathematics, eventually leading to the formalization of Zermelian set theory in first-order logic. In the last chapter Ferreirós describes the consolidation of axiomatic set theory, with prominent roles given to Fraenkel and von Neumann for their work on axiomatization, to Zermelo for the cumulative hierarchy, and to Gödel for his relative consistency results. It is a considerable credit to Ferreirós that the emergence of first-order axiomatic set theory unfolds smoothly and plausibly in his account through mathematical results, particularly those of Gödel's that affirmed its efficacy and superiority.

Much of history is in the details, and much of good history is in tone, emphasis, and the drawing together of different strands. Without being Whiggish, Ferreirós is never far from major synthetic themes, themes that in the main work toward balance and pull back from excesses of emphasis of some recent histories. The main criticism of the book would have to be on matters of broad interpretation. Yet, Ferreirós brings a considerable mathematical acuity and expertise to his enterprise, and this makes his viewpoints the more compelling.

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Marcus Kracht. ***Tools and techniques in modal logic.*** Studies in logic and the foundations of mathematics, vol. 142. Elsevier, Amsterdam etc. 1999, xiii + 559 pp.

Because modal logic became an independent branch of logic with its own internal programs only in the second half of the twentieth century, it is not surprising that only a few textbooks—and actually no advanced ones—in modal logic were available. The late 1990's have filled this gap by producing a number of advanced textbooks in the field. The book under review is one of these, and it gives an account of the present state of affairs in modal logic.

The book consists of three parts divided over ten chapters. The first part, *The fundamentals*, consists of three chapters. In Chapter 1 the author gives an account of the logical, algebraic, and computational background needed for reading the book. Chapters 2 and 3 are devoted to the basics of modal logic. Modal algebras and Kripke frames are introduced, and several important constructions are performed on them. Issues of completeness, correspondence, the finite model property, and decidability are raised, and the general structure of the lattice of modal logics is studied. Special attention is paid to a number of important modal logics, as well as to the classes of subframe logics and weakly transitive logics. The first part ends with a detailed discussion of a constructive reduction, Craig's interpolation, and Beth's definability properties.

Part 2, *The general theory of modal logic*, consists of four chapters. Chapter 4 discusses universal algebraic, topological, and some of the category-theoretical tools which play an important role in understanding the structure of the lattice of modal logics. Modal duality theory is developed on the basis of the Stone representation. Chapter 5 deals with correspondence theory. Famous results by Sahlqvist, Fine, and van Benthem are proved. In Chapter 6 the author studies transfer theorems between polymodal and monomodal logics. Fusions of modal logics are introduced, and it is proved that many important properties of logics transfer to their fusion. Special attention is paid to the Thomason simulation; it is used to show that the lattice of bimodal logics is isomorphic to an interval of the lattice of monomodal logics. Transfer of logical properties under that isomorphism is also discussed in great detail. Chapter 7 studies the global structure of the lattice of modal logics. The main technical tool is that of splitting, coming from lattice theory. A general splitting theorem is proved, and Blok's famous results on covers and degree of incompleteness are derived. The duality theory for upper continuous lattices is developed, and this is used for obtaining a clearer picture of several "good" sublattices of the lattice of modal logics. Finally, the lattice of tense logics is studied.

The third and final part, *Case studies*, consists of three chapters. Chapter 8 studies extensions of **K4**. It contains a detailed study of subframe logics, cofinal subframe logics, logics of finite width, and pre-tabular logics over **K4** together with Zakharyaschev's canonical formulas. The author proves a large number of (pre-)completeness and (pre-)decidability results for these logics. Chapter 9 studies logics of bounded alternativity. The lattice structure of extensions of $\mathbf{K.alt}_1$ is exhaustively described, and the complexity of the structure of the lattice of extensions of the fusion of $\mathbf{K.alt}_1$ with itself is explored. Finally, a technique is developed for showing that for many properties of logics, the problem whether a logic has this property or not, is undecidable. Chapter 10 deals with **PDL**—propositional dynamic logic. The finite model property and decidability for **PDL** and its various extensions and fragments are proved, and the famous interpolation issue is discussed in great detail. At the end of the book, the author gives a rather extensive list of references.

As a whole, the book deserves attention from specialists in the field. Of particular interest are Chapters 4, 7, and 8, which for the first time give a rather profound algebraic insight into the global structure of the lattice of modal logics. Another novelty is Chapter 6, which is a first attempt to give a rather extensive account of connections between monomodal and polymodal logics. Chapter 10 also deserves attention as a first systematic study of propositional dynamic logic.

The book has some shortcomings. There is insufficient motivation for anyone but experienced researchers. Little attempt is made to make the ideas transparent, and there are abuses of notations (definitions) which make several parts of the book quite difficult to follow. Moreover, several theorems are proved in an overly complicated way, particularly in Chapter 5. Finally, the carelessness of the author's style frequently leads to mistakes, which are scattered throughout the book.

In spite of these drawbacks, I think specialists as well as very well-prepared students will enjoy reading the book, because it covers interesting problems of modal logic. The author has succeeded in showing that the use of mathematical (algebraic and topological) tools provides an excellent means for deeper understanding of the theory.

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A. S. Troelstra and H. Schwichtenberg. ***Basic proof theory.*** Second edition of JSL LXIII 1605. Cambridge tracts in theoretical computer science, no. 43. Cambridge University Press, Cambridge, New York, etc., 2000, xii + 417 pp.

The present review of this second edition covers only the changes and additions to the first edition (JSL LXIII 1605). The main additions are the inclusion of solutions to some exercises, a section on proof systems with "local rules" (as opposed to the rules in natural deduction, where assumption discharge is non-local), more material on multi-succedent non-classical sequent calculi, applications of cut-elimination to predicate logic with equality and the intuitionistic theory of apartness, work of Negri and von Plato (this Bulletin, vol. 4 (1998), pp. 418–435) on cut-elimination for applied sequent calculi and (unpublished) on generalised elimination rules, an example (based on ideas of Statman) of a sequence of propositional sequents for which avoidance of cuts exponentially increases proof size, and recent bibliographic data. Changes include correction of many errors, with a proper treatment of cut-elimination (correcting the earlier confusion of two techniques), an extended exposition of some basic ideas about natural deduction systems and their relation to sequent calculi, an improved treatment of E-logic (i.e., "free logic"), an elegant new proof (from the thesis by Ralph Matthes) of strong normalisation for second-order lambda calculus, the replacement of "contracts to" by "converts to" (avoiding confusion with the "Contraction" rules of sequent calculi), and the mysterious renaming of Kees Doets (author of ***From logic to logic programming***, The MIT Press, 1994) as "D. H. C."

The book's strengths remain undiminished, augmented by extra explanations and good coverage of recent work. Availability of this graduate-level textbook in paperback is especially welcome.

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Ulrich Kohlenbach. *Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals*. ***Archive for mathematical logic***, vol. 36 (1996), pp. 31–71.

It is well known that even for mathematically rather weak theories, the provably recursive functions (i.e., the number-theoretic functions f having a definition $f(u) := \mu w\, A_0(u, w)$ where A_0 is a quantifier-free formula such that $\forall u \in \mathbb{N}\, \exists w \in \mathbb{N}\, A_0(u, w)$ is provable in the theory) may grow extremely fast. The paper under review introduces a variety of theories which in contrast to this general observation have provably recursive functions with a very low rate of growth, but nevertheless are strong enough to formalize substantial parts of classical analysis. The paper not only provides upper bounds for the provably recursive functions, but deals with the following more general and relevant situation: Many interesting theorems in analysis can be written in the form

$$\forall u \in U\, \forall v \in K\, \exists w \in \mathbb{N}\, A_0(u, v, w)$$

where U is a complete separable metric space, K is a compact metric space, and A_0 is quantifier-free (typically w will be a stopping number for an algorithm solving an approximation problem with a given accuracy). In the paper it is shown how to extract from a proof of such a formula a uniform bound Φu on w that does not depend on $v \in K$, i.e.,

$$\forall u \in U\, \forall v \in K\, \exists w \leqq \Phi u\, A_0(u, v, w).$$

The functional Φ will be of very low complexity, e.g. linear or a polynomial, depending on the theory in which the proof was carried out. The terms 'linear' and 'polynomial' refer to a standard representation of the elements of the metric space U by type-one objects, i.e., number-theoretic functions.

The theories considered are formulated in a language of finite types over $\mathbb{N}$ with equality on $\mathbb{N}$ and use classical logic. The basic theories G_nA^ω (Grzegorczyk arithmetic of level $n \geqq 1$) contain also some basic arithmetical relations and functions, constants for bounded minimization, bounded primitive recursion, and the branches A_i of the Ackermann function for $i \leqq n$. The axioms are all the true purely universal sentences with quantifiers over type $\leqq 2$, where 'true' refers to the full set-theoretic hierarchy of finite types. In particular G_nA^ω includes the axiom of quantifier-free induction, which can be written as a purely universal formula. The type-two functionals definable in G_nA^ω have a low rate of growth. For example for $n = 1, 2, 3$, they are bounded by linear, polynomial, elementary recursive functionals, respectively. By adding to G_nA^ω quantifier-free choice (AC-qf) and true axioms Γ of the form $\forall x^\delta\, \exists y \leq_\rho sx\, \forall z^\eta\, F_0(x, y, z)$ with $F_0(x, y, z)$ quantifier-free, one obtains theories in which substantial parts of classical analysis can be formalized.

A typical result of the paper, capturing and generalizing the situation discussed above, is the following: From a proof of

$$G_nA^\omega + \text{AC-qf} + \Gamma \vdash \forall u^1\, \forall k^0\, \forall v \leq_\tau tuk\, \exists w^0\, A_0(u, k, v, w)$$

one can extract a uniform bound Φ, definable in G_n^ω, such that

$$\forall u^1\, \forall k^0\, \forall v \leq_\tau tuk\, \exists w^0 \leqq \Phi uk\, A_0(u, k, v, w)$$

is true (note that the set $\{v \mid v \leq_\tau tuk\}$ represents the compact space K above). The result still holds if Γ contains axioms that are false in the set-theoretic hierarchy, but hold in a hierarchy of hereditarily majorizable functionals (a hierarchy introduced by Howard). A particularly interesting formula of this kind is an axiom F^- which is used to prove a uniform boundedness principle and a generalization of weak König's lemma.

The extraction of the uniform bounds is achieved by using a *monotone functional interpretation* whose advantages over Gödel's original functional interpretation are that the extracted functionals are simpler and that many important non-effective analytical principles, in particular those of the form Γ above, can be interpreted.

The paper contains many other related results in the same spirit, and in the subsequent paper *Relative constructivity*, ***The journal of symbolic logic***, vol. 63 (1998), pp. 1218–1238, Kohlenbach shows that many important non-effective analytical theorems can be proved even in G_2A^ω + AC-qf + Γ for suitable Γ, thus underpinning the relevance of the results of the paper reviewed here.

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PAUL ERDŐS, VANCE FABER, and JEAN LARSON. *Sets of natural numbers of positive density and cylindric set algebras of dimension 2*. ***Algebra universalis***, vol. 12 (1981), pp. 81–92.

JEAN A. LARSON. *The number of one-generated diagonal-free cylindric set algebras of finite dimension greater than two*. Ibid., vol. 16 (1983), pp. 1–16.

JEAN A. LARSON. *The number of finitely generated infinite cylindric set algebras of dimension two*. Ibid., vol. 19 (1984), pp. 377–396.

JEAN A. LARSON. *The number of one-generated cylindric set algebras of dimension greater than two*. ***The journal of symbolic logic***, vol. 50 (1985), pp. 59–71.

Some mathematicians (such as P. Erdős and A. Tarski) and some books (such as the Scottish book) are known for their problems. Such is the case for S. Ulam and his book, ***A collection of mathematical problems***, published in 1960 (JSL XXVII 368, republished as ***Problems in modern mathematics***, John Wiley and Sons, 1964). Many of the problems that Ulam put into his book were taken from the Scottish book, which he had recently translated from Polish and circulated privately. Ulam participated in the mid-twentieth century revival of algebraic logic by creating projective algebras in a 1946 paper with C. J. Everett (*Projective algebra I*, JSL XI 85). He included several problems about projective algebras in his book. This sequence of papers answers all of Ulam's questions.

Projective algebras are definitionally equivalent with diagonal-free cylindric algebras that have a distinguished atom satisfying various hypotheses (Theorem 5.2.4 of L. Henkin, J. D. Monk, A. Tarski, ***Cylindric algebras, Part II***, JSL LIII 651). Consequently these papers approach Ulam's problems through results about diagonal-free cylindric algebras.

The first paper presents a countable collection C of subsets of $\omega \times \omega$ that is not contained in any finitely generated diagonal-free cylindric set algebra of subsets of $\omega \times \omega$. C contains the ordering relation $\Delta = \{(n, m) : n < m\}$ and the "lines" $L_d = \{(n, n + d) : n \in \omega\}$. C generates a two-dimensional diagonal-free cylindric set algebra that is not isomorphic to a subalgebra of any finitely generated two-dimensional diagonal-free cylindric set algebra, and this algebra has the additional property that every one of its finitely generated subalgebras is contained in a two-generated two-dimensional diagonal-free cylindric set algebra. Similar statements hold for the projective algebra generated by C. Some computations involved in this construction use a result by Erdős, presented in the first section, concerning sets of natural numbers of positive density. At the end of the first paper the following answers are given to Ulam's problems (pp. 12–13), here in italics.

Given a countable class of sets in the plane, does there exist a finite number of sets which generate a projective algebra containing all sets of this countable class? Ulam notes that two plane sets can generate an infinite projective algebra (by an example attributed to David Nelson). Nevertheless, the answer is "no," because the projective algebra generated by C is not isomorphic to a subalgebra of any finitely generated projective algebra.

Does there exist a universal countable projective algebra, i.e., a countable projective algebra such that every countable projective algebra is isomorphic to some subalgebra of it? The answer is "no," because any such algebra can only have countably many subalgebras, and, by the second and fourth papers, there are continuum-many isomorphism types of projective algebras.

Is it true that, for every positive integer k, there exists a projective algebra generated by k sets in the plane and which is free in the sense that no relations exist between the generated sets except those that are true in every projective algebra? Can every projective algebra be obtained by a homomorphism of a free projective algebra? The answer is "no," because projective algebras cannot have non-trivial homomorphic images.

How many non-isomorphic projective algebras exist with k generators? The answer is $2^{\aleph_0}$. The second and fourth papers show that the number of non-isomorphic one-generated projective algebras is as large as possible, namely $2^{\aleph_0}$.

The second paper shows that for every $n \geqq 3$ there are continuum-many non-isomorphic one-generated n-dimensional algebras of these kinds: projective algebras, diagonal-free cylindric set algebras, diagonal-free cylindric algebras, and cylindric algebras. The algebras all have subsets of ω^n as elements. The only required operations (besides the Boolean ones) are the n cylindrifications. All the algebras are naturally two-generated by the set Γ consisting of sequences of natural numbers whose minimum coordinate is even, together with a set of the form $N \times (\omega^{n-1})$ for some $N \subseteq \omega$. If each such algebra is denoted by $B(N)$, then $B(N)$ and $B(M)$ are isomorphic iff N differs from M or the complement of M by a finite set. The fourth paper extends the results of the third to cylindric set algebras, provides a uniform proof for all the results, and—with suitable modifications to the hypotheses—also extends the results to the two-dimensional case. The crucial change is the replacement of Γ by the set S consisting of those sequences of natural numbers in which the second term is one more than the first. The second paper concentrates on the structure of the algebras, including a normal form for their elements, while the fourth paper looks instead at the possible forms of isomorphisms between the algebras. The fourth paper also shows that the numbers of isomorphism types of various other sorts of cylindric algebras are as large as possible.

The third paper contains a remarkable result: there are exactly seven isomorphism types of infinite one-generated two-dimensional diagonal-free cylindric set algebras, and representatives of those classes may be obtained as follows. A two-dimensional diagonal-free cylindric set algebra is a Boolean algebra of subsets of $U \times V$, for some sets U and V. The set $U \times V$ is called the Boolean unit of the algebra. To specify a one-generated two-dimensional diagonal-free cylindric set algebra, it suffices to give its unit and a generator. Define three units: $\Omega = \omega \times \omega$, $\Omega_0 = \omega \times (\omega \cup \{-1\})$, and $\Omega_1 = (\omega \cup \{-1\}) \times \omega$. Define three generators: $G(\leqq) = \{(n,m) : n \leqq m\}$, $G(\geqq) = \{(n,m) : n \geqq m\}$, and $G(L) = \{(n,m) : \min(n,m) \text{ is even}\}$. Define seven algebras by specifying for each a unit and a generator—$A(\leqq)$ has unit Ω and generator $G(\leqq)$, $A_0(\geqq)$ has unit Ω_0 and generator $G(\geqq)$, $A_1(\leqq)$ has unit Ω_1 and generator $G(\leqq)$, $A(\geqq)$ has unit Ω and generator $G(\geqq)$, $A(L)$ has unit Ω and generator $G(L)$, $A_0(L)$ has unit Ω_0 and generator $G(L)$, and $A_1(L)$ has unit Ω_1 and generator $G(L)$. Then every one-generated two-dimensional diagonal-free cylindric set algebra is isomorphic to either $A(\leqq)$, $A_0(\geqq)$, $A_1(\leqq)$, $A(\geqq)$, $A(L)$, $A_0(L)$, or $A_1(L)$.

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Uri Abraham. *Aronszajn trees on $\aleph_2$ and $\aleph_3$*. ***Annals of pure and applied logic***, vol. 24 (1983), pp. 213–230.

James Cummings and Matthew Foreman. *The tree property*. ***Advances in mathematics***, vol. 133 (1998), pp. 1–32.

Menachem Magidor and Saharon Shelah. *The tree property at successors of singular cardinals*. ***Archive for mathematical logic***, vol. 35 (1996), pp. 385–404.

The papers under review concern the relative consistency of the tree property on the $\aleph_n$'s and on $\aleph_{\omega+1}$. We begin with some definitions (which can also be found in these papers). Throughout, we work in ZFC.

Recall that a tree $\mathfrak{T}$ is a partial ordering in which the set of predecessors of any $a \in \mathfrak{T}$ is well ordered. $a \in \mathfrak{T}$ has height α if the order type of the set of predecessors of a is α. $\mathfrak{T}_\alpha$, the αth level of $\mathfrak{T}$, is the set of all points in $\mathfrak{T}$ of height α. The height of $\mathfrak{T}$ is the supremum of the heights of the non-empty levels. A branch of $\mathfrak{T}$ is a downward-closed linearly ordered subset of $\mathfrak{T}$.

For κ a cardinal, a κ-tree $\mathfrak{T}$ is a tree of height κ in which every level has size $< \kappa$. A κ-tree $\mathfrak{T}$ is an Aronszajn tree if it has no branch of length κ. κ has the tree property if there are no Aronszajn trees of height κ.

The following results provide both background and motivation to the theorems under consideration. All are stated as well in the paper by Cummings and Foreman: (1) $\aleph_0$ satisfies the tree property (König, *Sur les correspondances multivoques des ensembles*, ***Fundamenta mathematicae***, vol. 8 (1926), pp. 114–134). (2) $\aleph_1$ carries an Aronszajn tree (Aronszajn, found in Kurepa, *Ensembles ordonnés et ramifiés*, ***Publications mathématiques de l'Université de Belgrade***, vol. 4 (1935), pp. 1–138). (3) If $|[\kappa]^{<\kappa}| = \kappa$, then κ^+ carries an Aronszajn tree (Specker, *Sur un problème de Sikorski*, ***Colloquium mathematicum***, vol. 2 (1949), pp. 9–12). (4) The theories "ZFC + there is a weakly compact cardinal" and "ZFC + $\aleph_2$ satisfies the tree property" are equiconsistent (Mitchell, *Aronszajn trees and the independence of the transfer property*, ***Annals of mathematical logic***, vol. 5 (1972), pp. 21–46). (5) If κ is strongly inaccessible, then κ has the tree property iff κ is weakly compact.

Given Mitchell's result, it is natural to wonder how far this can be extended. In fact, Mitchell himself asks in his paper if it is possible, starting from a model containing two weakly compact cardinals, to get a model in which both $\aleph_2$ and $\aleph_3$ satisfy the tree property.

Mitchell's question is addressed by Abraham in his paper. In it, he demonstrates that it is possible, starting from a model containing cardinals $\kappa < \lambda$ such that κ is supercompact and λ is weakly compact, to obtain a model in which both $\aleph_2$ and $\aleph_3$ satisfy the tree property. His proof employs a two-step iteration, in which the forcing ensures that κ becomes $\aleph_2$, λ becomes $\aleph_3$, $2^{\aleph_0} = \aleph_2$, $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$, and the tree property is preserved for both κ and λ. Mitchell's argument from his 1972 paper cited above is used to show that λ retains the tree property, and the supercompactness of κ is used to show that κ retains the tree property.

It is no accident that very strong hypotheses are needed to obtain Abraham's result. It is a theorem of Magidor (found in Abraham's paper) that if $\aleph_2$ and $\aleph_3$ both satisfy the tree property, then there is an inner model with a measurable cardinal. As mentioned by Cummings and Foreman in their paper, this result has been improved by Foreman and Magidor, who have shown that if $\aleph_2$ and $\aleph_3$ both satisfy the tree property, then there is an inner model with a strong cardinal. Thus, Mitchell's question has a negative answer, since the consistency strength of having $\aleph_2$ and $\aleph_3$ simultaneously satisfy the tree property is much stronger than the existence of two weakly compact cardinals.

The paper by Cummings and Foreman contains two theorems. The first of these theorems extends Abraham's result to the remainder of the $\aleph_n$'s and was discovered independently by Cummings and by Foreman. (The constructions used were slightly different from one another, and the paper contains the one due to Cummings.) Specifically, starting from a model containing an ω sequence $\langle \kappa_n : n < \omega \rangle$ of supercompact cardinals, the authors force and obtain a model in which each κ_n becomes $\aleph_{n+2}$, $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$, and the κ_n's retain the tree property. The proof employs an ω-stage iteration of a suitable version of Abraham's forcing, with supercompactness used to overcome the technical problems encountered.

The second of these theorems is due to Foreman and obtains the tree property at the double successor of a singular strong limit cardinal κ of cofinality ω. (A modification of the proof given so that $\aleph_\omega$ is the strong limit cardinal and $\aleph_{\omega+2}$ has the tree property is hinted at.) By Specker's result mentioned above, the singular cardinals hypothesis will of necessity have to fail at κ. Specifically, starting from cardinals $\kappa < \lambda$ such that κ is supercompact and λ is weakly compact, a model is constructed in which κ becomes a singular strong limit cardinal of cofinality ω, κ^+ is preserved, $2^\kappa = \kappa^{++} = \lambda$, and λ has the tree property. The proof first makes κ indestructibly supercompact under κ-directed closed forcing (as done by Laver in his paper *Making the supercompactness of κ indestructible under κ-directed closed forcing*, ***Israel journal of mathematics***, vol. 29 (1978), pp. 385–388), and then uses a forcing iteration that combines the partial ordering for making $2^\kappa = \lambda$ with both Prikry forcing and different versions of the forcing for adding a single Cohen subset to κ^+. A fine analysis of this iteration then shows that the final model constructed is as desired.

The paper by Magidor and Shelah also contains two theorems. The first is a ZFC theorem which states that if λ is a singular limit of strongly compact cardinals, then λ^+ satisfies the tree property. The second, the main theorem of the paper, states that it is consistent, relative to technical hypotheses intermediate in consistency strength between a huge and a 2-huge cardinal, for $\aleph_{\omega+1}$ to satisfy the tree property.

The proof of the first of these theorems uses the fact that λ^+ is the successor of a singular limit of strongly compact cardinals to construct a branch of length λ^+ through any λ^+ tree. The proof of the second of the theorems is more involved. It employs the hugeness assumptions to show that a certain preparatory forcing preserves enough hugeness so that the relevant collapsing partial ordering which creates the $\aleph_{\omega+1}$ of the final model and the cardinal μ which becomes the $\aleph_{\omega+1}$ of the final model satisfy the desired technical properties required to complete the argument. There is no problem in generalizing the proof given to larger singular cardinals, regardless of cofinality.

All three of these papers are interesting, well written, and represent significant contributions to the theory of large cardinals. They also leave open many tantalizing questions. Some of these are as follows: (1) (Cummings and Foreman) Is it consistent for $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ to satisfy the tree property simultaneously? (2) (Cummings and Foreman) Is it consistent for $\aleph_{\omega+1}$ to satisfy the tree property with $2^{\aleph_\omega} > \aleph_{\omega+1}$? (In Magidor and Shelah's model, $2^{\aleph_\omega} = \aleph_{\omega+1}$.) (3) (Cummings and Foreman) Is it consistent for $\aleph_n$ for $2 \leqq n < \omega$ and for $\aleph_{\omega+1}$ to satisfy the tree property simultaneously? (4) In general, is it consistent for every successor cardinal $\kappa \neq \aleph_1$ to satisfy the tree property simultaneously?

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FRED APPENZELLER. *An independence result in quadratic form theory: infinitary combinatorics applied to ε-Hermitian spaces*. ***The journal of symbolic logic***, vol. 54 (1989), pp. 689–699.

OTMAR SPINAS. *Linear topologies on sesquilinear spaces of uncountable dimension*. ***Fundamenta mathematicae***, vol. 139 (1991), pp. 119–132.

JAMES E. BAUMGARTNER, MATTHEW FOREMAN, and OTMAR SPINAS. *The spectrum of the Γ-invariant of a bilinear space*. ***Journal of algebra***, vol. 189 (1997), pp. 406–418.

JAMES E. BAUMGARTNER and OTMAR SPINAS. *Independence and consistency proofs in quadratic form theory*. ***The journal of symbolic logic***, vol. 56 (1991), pp. 1195–1211.

OTMAR SPINAS. *Iterated forcing in quadratic form theory*. ***Israel journal of mathematics***, vol. 79 (1992), pp. 297–315.

OTMAR SPINAS. *Cardinal invariants and quadratic forms*. ***Set theory of the reals***, edited by Haim Judah, Israel mathematical conference proceedings, vol. 6, Gelbart Research Institute for Mathematical Sciences, Bar-Ilan University, Ramat-Gan 1993, distributed by the American Mathematical Society, Providence, pp. 563–581.

SAHARON SHELAH and OTMAR SPINAS. *Gross spaces*. ***Transactions of the American Mathematical Society***, vol. 348 (1996), pp. 4257–4277.

These papers apply set theory to investigate the properties of a vector space E of infinite dimension over a field F equipped with a symmetric bilinear form $\Phi : E \times E \to F$; call such a space a bilinear space, for short. (There is a well-known correspondence between symmetric bilinear forms and quadratic forms.) If U is a subspace of E, the orthogonal complement of U, denoted $U^\perp$, is defined to be $\{x \in E : \Phi(u, x) = 0 \text{ for all } u \in U\}$.

The first three papers in the list deal with a certain invariant, $\Gamma(E)$, of a bilinear space (E, Φ) which was introduced by Appenzeller to measure how far the space is from being an orthogonal sum of subspaces of smaller dimension; in particular, in dimension $\aleph_1$ it measures how close E is to having an orthogonal basis. (The invariant was first introduced in the context of Abelian group theory by the reviewer.) If E has dimension κ, a regular cardinal, the invariant takes values in the Boolean algebra $\mathcal{P}(\kappa)/\mathcal{F}_\kappa$, where $\mathcal{F}_\kappa$ is the club filter on κ. The spectrum of the Γ-invariant in dimension κ is defined to be the range of Γ restricted to bilinear spaces of dimension κ (over a given field F). Appenzeller shows that in dimension ω_1, the spectrum of Γ is the full Boolean algebra $\mathcal{P}(\omega_1)/\mathcal{F}_{\omega_1}$; for arbitrary dimension he produces many elements in the spectrum, but leaves open the question of what the range of Γ can be.

In the second paper, Spinas shows that for $\kappa > \omega_1$, the spectrum of Γ can never be all of $\mathcal{P}(\kappa)/\mathcal{F}_\kappa$. He introduces a condition that is necessary for an element e of $\mathcal{P}(\kappa)/\mathcal{F}_\kappa$ to be in the range of Γ; this condition, which is given the name "heavy" in *The spectrum of the Γ-invariant of a bilinear space* is as follows: e is represented by a subset S of κ such that for every $\mu \in S$ of cofinality $\geqq \omega_1$, $S \cap \mu$ contains a club in μ. The latter paper deals with the question whether in dimension ω_2 this necessary condition is sufficient for an element of

$\mathcal{P}(\omega_2)/\mathcal{F}_{\omega_2}$ to be in the spectrum of Γ. First, it is proved that CH implies that every heavy element is in the spectrum of Γ in dimension ω_2. The proof introduces a combinatorial principle (the existence of a coherent ladder system) which resembles Jensen's $\square_{\omega_1}$-principle but is implied by CH. The same principle is forced in a model of $\neg$CH, so it is consistent with $\neg$CH that the spectrum is as large as it can be in dimension ω_2. Finally it is proved that in a variation of Mitchell's model (W. Mitchell, *Aronszajn trees and the independence of the transfer property*, ***Annals of mathematical logic***, vol. 5 (1972), pp. 21–46), where a weakly compact cardinal is collapsed to ω_2, there are heavy elements that are not in the spectrum of Γ in dimension ω_2.

The other papers deal with another question about bilinear spaces, namely the existence of Gross spaces. A bilinear space E is called a Gross space (or a strong Gross space) if it is of uncountable dimension and every infinite dimensional subspace U satisfies $\dim(U^{\perp}) < \dim(E)$ (or $\dim(U^{\perp}) \leqq \aleph_0$, respectively). Obviously a Gross space of dimension $\aleph_1$ is a strong Gross space. These spaces are named after Herbert Gross who, with E. Ogg (*Quadratic spaces with few isometries*, ***Commentarii mathematici Helvetici***, vol. 48 (1973), pp. 511–519), investigated them and showed that strong Gross spaces exist over any uncountable field F. Later it was shown that CH implies that (strong) Gross spaces of dimension $\aleph_1$ exist over any finite or countable field (cf. W. Baur and H. Gross, *Strange inner product spaces*, ***Commentarii mathematici Helvetici***, vol. 52 (1977), pp. 491–495). The dimension of a strong Gross space over F is at most $|F|^{\aleph_0}$. In the paper by Baumgartner and Spinas, the Baur–Gross result is generalized by showing that MA implies the existence of Gross spaces of dimension $2^{\aleph_0}$ over any finite or countable field. On the other hand, it is proved consistent with $\neg$CH that Gross spaces of dimension $2^{\aleph_0}$ over a finite or countable field do not exist.

In the same paper the relevance of certain cardinal invariants is introduced. The results proved there are improved and complemented in the papers in the ***Israel journal of mathematics*** and ***Transactions of the American Mathematical Society***. This review will summarize some of the results, for spaces over finite or countable fields. (A comprehensive survey is provided in *Cardinal invariants and quadratic forms*.) The key invariants are $\mathfrak{b}$, the minimum size of an unbounded (under eventual dominance) family in ${}^{\omega}\omega$; $\mathfrak{p}$, the minimum size of a family of infinite subsets of ω that has the finite intersection property but has no infinite pseudo-intersection; and $\mathfrak{s}$, the minimum size of a splitting family. It is known that $\aleph_1 \leqq \mathfrak{p} \leqq \mathfrak{b} \leqq 2^{\aleph_0}$ and $\mathfrak{p} \leqq \mathfrak{s}$. In these papers it is proved that $\mathfrak{b} = \aleph_1$ implies that strong Gross spaces exist over any infinite field and that $\mathfrak{p} > \aleph_1$ implies that they do not exist over any finite or countable field. Moreover, $\mathfrak{s} > \aleph_1$ implies that strong Gross spaces do not exist over any finite field. On the other hand, $\aleph_1 = \mathfrak{p} < \mathfrak{b}$ does not decide the question of existence. Moreover, it is consistent with $\mathfrak{s} = \aleph_1$ and with $\mathfrak{b} = \aleph_1$ that there are no strong Gross spaces over finite fields.

The paper with Shelah, *Gross spaces*, also investigates generalizations of the notion of the Gross property. A space E is called a λ-Gross space if for all subspaces U of dimension $\geqq \lambda$, $\dim(U^{\perp}) < \dim(E)$. The dimension of a λ-Gross space over F is at most $|F|^{\lambda}$; the best possible result in ZFC is proved: over every field of uncountable cardinality λ there is a λ-Gross space of dimension λ^{+} and a λ^{+}-Gross space of dimension λ^{++}.

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