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GROTHENDIECK RINGS OF $\mathbb{Z}$-VALUED FIELDS

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Abstract. We prove the triviality of the Grothendieck ring of a $\mathbb{Z}$-valued field K under slight conditions on the logical language and on K. We construct a definable bijection from the plane K^2 to itself minus a point. When we specialize to local fields with finite residue field, we construct a definable bijection from the valuation ring to itself minus a point.

At the Edinburgh meeting on the model theory of valued fields in May 1999, Luc Bélair posed the question of whether there is a definable bijection between the set of p-adic integers and the set of p-adic integers with one point removed. At the same meeting, Jan Denef asked what is the Grothendieck ring of the p-adic numbers, as did Jan Krajíček independently in [K]. A general introduction to Grothendieck rings of logical structures was recently given in [KS] and in [DL2, par. 3.7]. Calculations of non-trivial Grothendieck rings and related topics such as motivic integration can be found in [DL] and [DL2]. The logical notion of the Grothendieck ring of a structure is analogous to that of the Grothendieck ring in the context of algebraic K-theory and has analogous elementary properties (see [S]). Here we recall the definition.

DEFINITION 1. Let $\mathcal{M}$ be a structure and $\mathcal{Def}(\mathcal{M})$ the set of definable subsets of M^n for every positive integer n. For any $X, Y \in \mathcal{Def}(\mathcal{M})$, write $X \cong Y$ if and only if there is a definable bijection (an isomorphism) from X to Y. Let F be the free abelian group whose generators are isomorphism classes $\lfloor X \rfloor$ with $X \in \mathcal{Def}(\mathcal{M})$ (so $\lfloor X \rfloor = \lfloor Y \rfloor$ if and only if $X \cong Y$) and let E be the subgroup generated by all expressions $\lfloor X \rfloor + \lfloor Y \rfloor - \lfloor X \cup Y \rfloor - \lfloor X \cap Y \rfloor$ with $X, Y \in \mathcal{Def}(\mathcal{M})$. Then the Grothendieck group of $\mathcal{M}$ is the quotient group F/E. Write $[X]$ for the image of $X \in \mathcal{Def}(\mathcal{M})$ in F/E. The Grothendieck group has a natural structure as a ring with multiplication induced by $[X] \cdot [Y] = [X \times Y]$ for $X, Y \in \mathcal{Def}(\mathcal{M})$. We call this ring the Grothendieck ring $K_0(\mathcal{M})$ of $\mathcal{M}$.

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It is easy to see that the above questions are related: the Grothendieck ring is trivial if and only if there is a definable bijection between M^k and itself minus a point for some k, which happens if and only if the Grothendieck group is trivial. Moreover, if we find such a k then we have for any $X \in \mathcal{Def}(\mathcal{M})$ a definable bijection from the disjoint union of $M^k \times X$ and X to $M^k \times X$; if there is a definable injection from M^k into X we find a definable bijection from X to itself minus a point.

In this paper we answer the questions posed by Bélair and Denef. Furthermore, we prove the triviality of the Grothendieck ring of any $\mathbb{Z}$-valued field which satisfies some slight conditions and give in this general setting an explicit bijection from the plane to itself minus a point. For the fields $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ we explicitly construct a definable bijection from the valuation ring to itself minus a point.

Dave Marker independently produced a definable bijection from $\mathbb{Z}_p$ to $\mathbb{Z}_p \setminus \{0\}$, after it was noticed by Lou van den Dries that its existence followed from unpublished notes of the second author. The first author has proved further that there is a definable bijection between any two definable sets in the p-adics if and only if they have the same dimension. This will appear in a later paper. We thank the referee for encouraging us to present these results in greater generality than had been our original intention.

Fix a $\mathbb{Z}$-valued field K, that is, a field with a valuation $v\colon K^\times \to Z$ to an ordered group Z which is elementarily equivalent to the integers in the Presburger language. Let $R = \{ x \in K \mid v(x) \geq 0 \}$ be the valuation ring, $R^* = R \setminus \{0\}$ and $\bar{K} = R/m$ the residue field, with m the maximal ideal of R and natural projection $R \to \bar{K}\colon x \to \bar{x}$. An angular component map is a homomorphism ac: $K^\times \to \bar{K}^\times$ such that $ac(x) = \bar{x}$ if $v(x) = 0$. We extend ac to a map $ac\colon K \to \bar{K}$ by putting $ac(0) = 0$ (for the existence of angular component maps, see [P] and [B]).

DEFINITION 2. Let $\mathcal{L}$ be an extension of the language of rings with K as a model. We say that the structure $(K, \mathcal{L})$ satisfies condition $(*)$ if we can choose an angular component map ac and an $\mathcal{L}$-definable element $\pi \in R$ with $v(\pi) = 1$ and $ac(\pi) = 1$ such that the sets R and $R^{(1)} = \{ x \in R \mid ac(x) = 1 \}$ are $\mathcal{L}$-definable.

Notice that if condition $(*)$ is satisfied, the set

$$\{ (x, y) \in K^2 \mid v(x) \leq v(y) \}$$

is $\mathcal{L}$-definable by the formula $\exists z \in R\ (zx = y)$. A bijection $X \to Y$ with X, $Y \in \mathcal{Def}(K, \mathcal{L})$ with $\mathcal{L}$-definable graph will be called an isomorphism.

Let $X \subset K^m$ and $Y \subset K^n$ be definable sets, $m \geq n$. Let $X' = \{0\} \times X$ and $Y' = \{1\}^{m-n+1} \times Y$. Then we define the disjoint union $X \sqcup Y$ of X and Y up to isomorphism to be $X' \cup Y'$. We say that a set W is isomorphic to $X \sqcup Y$ if W is isomorphic to $X' \cup Y'$ and then obviously $[W] = [X] + [Y]$. If

$(K, \mathcal{L})$ satisfies condition $(*)$ then we can find $W \subset R^m$ with $W \cong X \sqcup Y$ as follows. The map $i: K \to R$ which sends x to πx if $v(x) \geq 0$ and to $1 + 1/x$ if $v(x) < 0$ is a definable injection. For $m = n = 1$, put $X'' = \pi.i(X)$ and $Y'' = 1 + \pi.i(Y)$. Then $X'' \cong X$, $Y'' \cong Y$ and $X'' \cap Y'' = \phi$, so $W = X'' \cup Y''$ is isomorphic to $X \sqcup Y$. For $m > 1$, use the same method in each coordinate.

PROPOSITION 1. *Let K be a $\mathbb{Z}$-valued field, which is a model for the language $\mathcal{L}$. If the structure $(K, \mathcal{L})$ satisfies condition $(*)$, then the following holds*:

(i) *The disjoint union of R and $R^{(1)}$ is isomorphic to $R^{(1)}$ and thus $[R] = 0$.*
(ii) *The disjoint union of two copies of R^{*2} is isomorphic to R^{*2} itself, and hence $[R^{*2}] = 0$.*

PROOF.
(i) The map

$$\{0\} \times R \cup \{1\} \times R^{(1)} \to R^{(1)} \colon \begin{cases} (0, x) \mapsto 1 + \pi x, \\ (1, x) \mapsto \pi x, \end{cases}$$

is easily seen to be an isomorphism as required. This yields in the Grothendieck ring $[R] + [R^{(1)}] = [R^{(1)}]$, so $[R] = 0$.

(ii) Define the sets

$$X_1 = \{ (x, y) \in R^{*2} \mid v(x) \leq v(y) \},$$
$$X_2 = \{ (x, y) \in R^{*2} \mid v(x) > v(y) \},$$

then X_1, X_2 form a partition of R^{*2}. The isomorphisms

$$\{0\} \times R^{*2} \to X_1 \colon (0, x, y) \mapsto (x, xy),$$
$$\{1\} \times R^{*2} \to X_2 \colon (1, x, y) \mapsto (\pi xy, y),$$

imply that $R^{*2} \sqcup R^{*2}$ is isomorphic to $X_1 \cup X_2 = R^{*2}$. It follows that $2[R^{*2}] = [R^{*2}]$, so $[R^{*2}] = 0$. Notice that the proof of (ii) does not use the full power of $(*)$, only that R is definable. ⊣

THEOREM 1. *Let K be a $\mathbb{Z}$-valued field, which is a model for the language $\mathcal{L}$. If the structure $(K, \mathcal{L})$ satisfies condition $(*)$, then the Grothendieck ring $K_0(K)$ is trivial and there exists an isomorphism from $R^2 \setminus \{(0, 0)\}$ to R^2.*

PROOF. Since $0 = [R] = [R^*] + [\{0\}]$ we have $[R^*] = -1$. Together with $0 = [R^{*2}] = [R^*]^2$ this yields $1 = 0$, so $K_0(K)$ is trivial.

Define the isomorphisms $\psi \colon R^2 \to \pi^3 R^2 \colon (x, y) \mapsto (\pi^3 x, \pi^3 y)$ and

$$\varphi_i \colon R^2 \to (\pi^i + \pi^3 R) \times (\pi^i + \pi^3 R) \colon (x, y) \mapsto (\pi^i + \pi^3 x, \pi^i + \pi^3 y)$$

for $i = 1, 2$.

Since clearly $\psi(R^* \times R^*) \cup \varphi_1(R^* \times R^*)$ is isomorphic to $R^{*2} \sqcup R^{*2}$, we can find by Proposition 1 (ii) an isomorphism

$$f_1 \colon \varphi_1(R^* \times R^*) \to \psi(R^* \times R^*) \cup \varphi_1(R^* \times R^*).$$

Define f_2 by

$$f_2\colon \psi(R \times R^*) \cup \varphi_2(R^{(1)} \times R^*) \to \varphi_2(R^{(1)} \times R^*):$$

$$\begin{cases} \psi(x,y) \mapsto \varphi_2(1+\pi x, y), \\ \varphi_2(x,y) \mapsto \varphi_2(\pi x, y). \end{cases}$$

Analogously, we can modify the function given in the proof of Proposition 1 (i) to get an isomorphism

$$f_3\colon \varphi_2(\{0\} \times R^{(1)}) \to \varphi_2(\{0\} \times R^{(1)}) \cup \psi(\{0\} \times R).$$

Finally, $g\colon R^2 \setminus \{(0,0)\} \to R^2$:

$$x \mapsto \begin{cases} f_1(x) & \text{if } x \in \varphi_1(R^* \times R^*), \\ f_2(x) & \text{if } x \in \psi(R \times R^*) \cup \varphi_2(R^{(1)} \times R^*), \\ f_3(x) & \text{if } x \in \varphi_2(\{0\} \times R^{(1)}), \\ x & \text{else}, \end{cases}$$

is the required isomorphism. $\dashv$

We give some examples for the conditions of Theorem 1 to be satisfied. Let $\mathcal{L}_{ac}$ be the language of rings with an extra constant symbol to denote π and a relation symbol to denote the set $R^{(1)}$. Let $\mathcal{L}_{ac,R}$ be the language $\mathcal{L}_{ac}$ with an extra relation symbol to denote R.

- Let K be a valued field with valuation to the integers $\mathbb{Z}$. Then we can define an angular component as follows. Choose $\pi \in K$ with $v(\pi) = 1$ and put $ac(x) = \overline{\pi^{-v(x)}x}$ for $x \neq 0$. Then clearly $ac(\pi) = 1$ and $(K, \mathcal{L}_{ac,R})$ satisfies condition $(*)$.
- Let K be a Henselian field with valuation to the integers $\mathbb{Z}$. Then R is already definable in the language of rings: if $\operatorname{char}(\bar{K}) \neq 2$ we have $R = \{\, x \in K \mid \exists y \in K, y^2 = 1 + \pi x^2 \,\}$ and if $\operatorname{char}(\bar{K}) = 2$ then we use the formula $\exists y \in K,\ y^3 = 1 + \pi x^3$ to define R. This implies that $(K, \mathcal{L}_{ac})$ satisfies condition $(*)$.
- For definability of the valuation ring in fields of rational functions within the language of rings, see [D] and [KR].

Now we specialize our attention to local fields with finite residue field.

THEOREM 2. *Let $K = \mathbb{F}_q((t))$ be the formal Laurent series over the finite field $\mathbb{F}_q$ and $\mathcal{L}_t$ the language of rings with a constant symbol to denote t. Then $K_0(K)$ is trivial and we have an isomorphism $R \to R^*$.*

PROOF. We first show that K satisfies condition $(*)$. Since K is a Henselian field, R is definable as shown above. For each $x \in \mathbb{F}_q$ we have $x^{q-1} = 1$, so we can define $R^{(1)}$ as

$$R^{(1)} = \left\{ x \in R \;\middle|\; \exists y \in R^*, \bigvee_{n=0}^{q-2} t^n y^{q-1} = x \right\},$$

again by Hensel's lemma.

By Theorem 1 we have an isomorphism $f : R^2 \to R^2 \setminus \{(0,0)\}$. For a Laurent series $H(t) \in K$ we have $H(t)^p = H(t^p)$. Consequently, the map

$$g : K^2 \to K : (x, y) \mapsto x^p + ty^p$$

is an injection from the plane into the line. We obtain the isomorphism

$$R \to R^* : x \mapsto \begin{cases} g \circ f \circ g^{-1}(x) & \text{if } x \in g(R^2), \\ x & \text{else.} \end{cases} \qquad \dashv$$

Now let $\mathbb{Q}_p$ be the field of p-adic numbers and K a fixed finite field extension of $\mathbb{Q}_p$. Choose an element π with $v(\pi) = 1$, then $ac(x) = \pi^{-v(x)}x$ mod (π) defines an angular component for $x \neq 0$. We work with $\mathcal{L}_\pi$, the language of rings with an extra constant symbol to denote π. For a definable set $X \subset K$ and $k \in \mathbb{N}_0$ we write

$$X^{(k)} = \{ x \in X \mid v(\pi^{-v(x)}x - 1) \geq k \},$$

which corresponds with our previous definition of $R^{(1)}$. The set R and each $X^{(k)}$ is definable by the same argument as in the proof of Theorem 2, so $(K, \mathcal{L}_\pi)$ satisfies condition $(*)$. We put $P_n = \{ x \in K^\times \mid \exists y \in K,\ y^n = x \}$ and $\bar{P}_n = P_n \cap R$. Recall that P_n is a subgroup of finite index in $K^\times$ for each n.

For convenience, we recall the following easy corollary of Hensel's Lemma.

Corollary 1. *Let $n > 1$ be a natural number. For each $k > v(n)$, and $k' = k + v(n)$ the function*

$$K^{(k)} \to P_n^{(k')} : x \mapsto x^n$$

is an isomorphism.

In the next proposition we exhibit some isomorphisms between definable sets.

Proposition 2. *Let K be a finite field extension of the p-adic numbers and $\mathcal{L}_\pi$ the language of rings with an extra constant symbol to denote π. Then we have*

(i) *for each $k > 0$, the union of two disjoint copies of $R^{(k)}$ is isomorphic to $R^{(k)}$;*
(ii) *the union of two disjoint copies of R^* is isomorphic to R^*.*

PROOF.
(i) **Case 1**: $p \neq 2$. The map $R^{(k)} \to \bar{P}_2^{(k)}: x \mapsto x^2$ is an isomorphism for each $k > 0$ by Corollary 1. By Hensel's Lemma, $R^{(k)} = \bar{P}_2^{(k)} \cup \pi \bar{P}_2^{(k)}$ is a partition. Hence the function

$$\{0\} \times R^{(k)} \cup \{1\} \times R^{(k)} \to R^{(k)}: \begin{cases} (0,x) \mapsto x^2, \\ (1,x) \mapsto \pi x^2, \end{cases}$$

is an isomorphism.

Case 2: $p = 2$. The map $R^{(k)} \to \bar{P}_3^{(k)}: x \mapsto x^3$ is an isomorphism by Corollary 1, and by Hensel's Lemma $R^{(k)} = \bar{P}_3^{(k)} \cup \pi \bar{P}_3^{(k)} \cup \pi^2 \bar{P}_3^{(k)}$ is a partition. Explicitly, we see that cubing and multiplying by 1, π or π^2 is an isomorphism from three disjoint copies of $R^{(k)}$ to $R^{(k)}$. First suppose that $k > v(2)$ and put $k' = k + v(2)$, then $R^{(k)} \to \bar{P}_2^{(k')}: x \mapsto x^2$ is an isomorphism by Corollary 1. By Hensel's lemma, we have a partition $R^{(k)} = \bigcup_{i=1}^{2^l} \alpha_i \bar{P}_2^{(k')}$ for some $l \in \mathbb{N}_0$. Thus we can say there are isomorphisms from $R^{(k)}$ to 2^l disjoint copies of $R^{(k)}$ and to three disjoint copies of $R^{(k)}$. Some arithmetic on the number of disjoint copies yields the required isomorphism for $k > v(2)$.

If $k \leq v(2)$ then $R^{(k)}$ admits a finite partition into parts of the form $\alpha R^{(v(2)+1)}$, with $v(\alpha) = 0$, and hence that the required isomorphism exists follows from property (i) for $R^{(v(2)+1)}$.

(ii) Since R^* admits a finite partition with parts of the form $\alpha R^{(1)}$ with $v(\alpha) = 0$, this follows from (i). ⊣

Now we give the solution of the problems raised by J. Denef and L. Bélair.

THEOREM 3. *Let K be a finite field extension of $\mathbb{Q}_p$ and $\mathcal{L}_\pi$ the language of rings with an extra constant symbol to denote π. Then $K_0(K) = 0$ and we have an isomorphism from R to itself minus a point.*

PROOF. The triviality of the Grothendieck ring follows from Theorem 1.
We write the isomorphism explicitly in the case $p \neq 2$. First let

$$W = 1 + \pi^2 R^* \cup \pi^2 R \cup \pi + \pi^2 R^{(1)}.$$

As in the proof of Proposition 2, we can write

$$R^* = \bigcup_{i=1}^{l} \alpha_i R^{(1)} = \bigcup_{i=1}^{l} (\alpha_i \bar{P}_2^{(1)} \cup \pi \alpha_i \bar{P}_2^{(1)})$$

as a partition for some $l \in \mathbb{N}_0$. Thus the function

$$f_1: \pi^2 R^* \cup 1 + \pi^2 R^* \to 1 + \pi^2 R^*: \begin{cases} \pi^2 \alpha_i x \mapsto 1 + \pi^2(\alpha_i x^2), \\ 1 + \pi^2 \alpha_i x \mapsto 1 + \pi^2(\pi \alpha_i x^2), \end{cases}$$

where $x \in R^{(1)}$, is a well-defined isomorphism. Modify the function given in the proof of Proposition 1 (i), to get

$$f_2\colon \pi^2 R \cup \pi + \pi^2 R^{(1)} \to \pi + \pi^2 R^{(1)}\colon \begin{cases} \pi^2 x \mapsto \pi + \pi^2(1 + \pi x), \\ \pi + \pi^2 x \mapsto \pi + \pi^2(\pi x). \end{cases}$$

Then the function

$$f\colon W \to W \setminus \{0\}\colon x \mapsto \begin{cases} f_1^{-1}(x) & \text{if } x \in 1 + \pi^2 R^*, \\ f_2(x) & \text{if } x \in \pi^2 R \cup \pi + \pi^2 R^{(1)}, \end{cases}$$

is an isomorphism. Finally,

$$g\colon R \to R^*\colon x \mapsto \begin{cases} f(x) & \text{if } x \in W \\ x & \text{if } x \notin W \end{cases}$$

is an isomorphism.

In the case $p = 2$, we know from Proposition 2 (i) that there is a function which plays the role of f_1. The rest is as above. $\dashv$

REMARK.

• The construction of the bijection $\mathbb{Z}_p \to \mathbb{Z}_p \setminus \{0\}$ also works for the field $\mathbb{F}_q((t))$ if $2+9$. If $2/9$ the proof of Proposition 2 (i) collapses since the index of the squares in $\mathbb{F}_q((t))^\times$ is infinite.

• The triviality of the Grothendieck ring of a structure $\mathcal{M}$ implies that every Euler characteristic on the definable sets is trivial. An Euler characteristic is a map $\chi\colon \mathcal{D}ef(\mathcal{M}) \to R_\chi$ with R_χ a ring, such that $\chi(X) = \chi(Y)$ if $X \cong Y$, $\chi(X \cup Y) = \chi(X) + \chi(Y)$ if $X \cap Y = \phi$ and $\chi(X \times Y) = \chi(X)\chi(Y)$. In general an Euler characteristic on $\mathcal{D}ef(\mathcal{M})$ factorizes through $\mathcal{D}ef(\mathcal{M}) \to K_0(\mathcal{M})\colon X \mapsto [X]$.

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