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REVIEWS

The Association for Symbolic Logic reviews selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Thomas Jech, Wolfram Pohlers, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VI 94" refers to the review beginning on page 94 in volume 6 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 125***5*** or 192***4*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

Logic, logic, and logic, by George Boolos, with introductions and afterword by John P. Burgess, edited by Richard Jeffrey, Harvard University Press, Cambridge, Mass., and London, 1998, ix + 443 pp.—therein:

JOHN P. BURGESS. *Introduction*. Pp. 3–11.

GEORGE BOOLOS. *The iterative conception of set*. A reprint of JSL L 544. Pp. 13–29.

GEORGE BOOLOS. *Reply to Charles Parsons' "Sets and classes."* Pp. 30–36.

GEORGE BOOLOS. *On second-order logic*. Pp. 37–53. (Reprinted from ***The journal of philosophy***, vol. 72 (1975), pp. 509–527.)

GEORGE BOOLOS. *To be is to be a value of a variable (or to be some values of some variables)*. A reprint of JSL LIV 616. Pp. 54–72.

GEORGE BOOLOS. *Nominalist Platonism*. A reprint of JSL LIV 616. Pp. 73–87.

GEORGE BOOLOS. *Iteration again*. Pp. 88–104. (Reprinted from ***Philosophical topics***, vol. 17 no. 2 (for 1989, pub. 1990), pp. 5–21.)

GEORGE BOOLOS. *Introductory note to Kurt Gödel's "Some basic theorems on the foundations of mathematics and their implications."* Pp. 105–119. (Reprinted from ***Collected works***, Volume III, ***Unpublished essays and lectures***, by Kurt Gödel, edited by Solomon Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons, and Robert M. Solovay, Oxford University Press, New York and Oxford 1995, pp. 290–304.)

GEORGE BOOLOS. *Must we believe in set theory?* Pp. 120–132. (Also in ***Between logic and intuition, Essays in honor of Charles Parsons***, edited by Gila Sher and Richard Tieszen, Cambridge University Press, Cambridge, New York, etc., 2000, pp. 257–268.)

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John P. Burgess. *Introduction*. Pp. 135–142.

George Boolos. *Gottlob Frege and The foundations of arithmetic*. Pp. 143–154.

George Boolos. *Reading the Begriffsschrift*. Pp. 155–170. (Also in ***Frege's philosophy of mathematics***, edited by William Demopoulos, Harvard University Press, Cambridge, Mass., and London, 1995, pp. 163–181; reprinted from ***Mind***, n.s. vol. 94 (1985), pp. 331–344.)

George Boolos. *Saving Frege from contradiction*. Pp. 171–182. (Also in ***Frege's philosophy of mathematics***, edited by William Demopoulos, Harvard University Press, Cambridge, Mass., and London, 1995, pp. 438–452; reprinted from ***Proceedings of the Aristotelian Society***, n.s. vol. 87 (1986/87), pp. 137–151.)

George Boolos. *The consistency of Frege's Foundations of arithmetic*. Pp. 183–201. (Also in ***Frege's philosophy of mathematics***, edited by William Demopoulos, Harvard University Press, Cambridge, Mass., and London, 1995, pp. 211–233; reprinted from ***On being and saying, Essays for Richard Cartwright***, edited by Judith Jarvis Thompson, The MIT Press, Cambridge, Mass., and London, 1987, pp. 3–20.)

George Boolos. *The standard of equality of numbers*. Pp. 202–219. (Also in ***Frege's philosophy of mathematics***, edited by William Demopoulos, Harvard University Press, Cambridge, Mass., and London, 1995, pp. 234–254; reprinted from ***Meaning and method, Essays in honor of Hilary Putnam***, edited by George Boolos, Cambridge University Press, Cambridge, New York, and Oakleigh, Victoria, 1990, pp. 261–277.)

George Boolos. *Whence the contradiction?* Pp. 220–236. (Reprinted from ***The Aristotelian Society supplementary volume LXVII***, London 1993, pp. 213–233.)

George Boolos. *1879?* Pp. 237–254. (Reprinted from ***Reading Putnam***, edited by Peter Clark and Bob Hale, Blackwell, Oxford and Cambridge, Mass., 1994, pp. 31–48.)

George Boolos. *The advantages of honest toil over theft*. A reprint of JSL LX 1009. Pp. 255–274.

George Boolos. *On the proof of Frege's theorem*. Pp. 275–290. (Reprinted from ***Benacerraf and his critics***, edited by Adam Morton and Stephen P. Stich, Philosophers and their critics, no. 8, Blackwell Publishers, Oxford and Cambridge, Mass., 1996, pp. 143–159.)

George Boolos. *Frege's theorem and the Peano postulates*. Pp. 291–300. (Reprinted from ***The bulletin of symbolic logic***, vol. 1 (1995), pp. 317–326.)

George Boolos. *Is Hume's principle analytic?* A reprint of JSL LXIII 1598. Pp. 301–314.

George Boolos and Richard G. Heck, Jr. *Die Grundlagen der Arithmetik, §§82–83*. A reprint of BSL VI 498. Pp. 315–338.

George Boolos. *Constructing Cantorian counterexamples*. Pp. 339–341. (Reprinted from ***Journal of philosophical logic***, vol. 26 (1997), pp. 237–239.)

John P. Burgess. *Introduction to Part III*. Pp. 345–353.

George Boolos. *Zooming down the slippery slope*. Pp. 354–364. (Reprinted from ***Noûs***, vol. 25 (1991), pp. 695–706.)

George Boolos. *Don't eliminate cut*. Pp. 365–369. (Reprinted from ***Journal of philosophical logic***, vol. 13 (1984), pp. 373–378.)

George Boolos. *The justification of mathematical induction*. Pp. 370–375. (Reprinted from ***PSA 1984***, Volume 2, ***Symposia***, edited by Peter D. Asquith and Philip Kitcher, Philosophy of Science Association, East Lansing 1985, pp. 469–475.)

George Boolos. *A curious inference*. Pp. 376–382. (Reprinted from ***Journal of philosophical logic***, vol. 16 (1987), pp. 1–12.)

George Boolos. *A new proof of the Gödel incompleteness theorem*. Pp. 383–388. (Reprinted from ***Notices of the American Mathematical Society***, vol. 36 (1989), pp. 388–390 and 676.)

George Boolos. *On "seeing" the truth of the Gödel sentence*. Pp. 389–391. (Reprinted from ***Behavioral and brain sciences***, vol. 13 (1990), pp. 655–656).

George Boolos. *Quotational ambiguity*. Pp. 392–405. (Reprinted from ***On Quine, New essays***, edited by Paolo Leonardi and Marco Santambrogio, Cambridge University Press, Cambridge, New York, and Oakleigh, Victoria, 1995, pp. 283–296.)

George Boolos. *The hardest logical puzzle ever*. Pp. 406–410. (Reprinted from ***The Harvard review of philosophy***, vol. 6 (1996), pp. 62–65. English version of *L'indovinello più difficile del mondo*, Italian translation by Massimo Piatteli-Palmarini, ***La repubblica***, April 16, 1992, pp. 36–37.)

George Boolos. *Gödel's second incompleteness theorem explained in words of one syllable*. Pp. 411–413. (Reprinted from ***Mind***, n.s. vol. 103 (1994), pp. 1–3).

John P. Burgess. *Afterword*. Pp. 415–424.

George Boolos died in his intellectual prime. At the time of his demise, one major component of his work, the study of provability logic, had been taken to fruition; ***The logic of provability*** (JSL LX 1316) is the definitive work on the subject. Other projects were in a less advanced state. In such cases, there is the worry that the author's ideas, presented in scattered journals and never fully developed, will fail to be fully appreciated. Richard Jeffrey and John Burgess have done much to forestall this possibility by assembling Boolos's work on topics other than provability logic, and other than the textbook ***Computability and logic*** (JSL XLII 585) (co-authored with Jeffrey), into a single, skillfully and lovingly edited, volume.

The book is divided into three parts, the first dealing with the foundations of set theory and with second-order logic.

Deeply troubled by the set-theoretic paradoxes, Boolos intended his work on set theory to dispel the notion that the Zermelo axioms are directed by no more focused an aim than to preserve classical mathematics while avoiding antinomy. Although it is not apparent from what Zermelo says, the axioms describe (as Gödel was the first to realize clearly) a unified and compelling picture of the structure of the set-theoretic universe. To make the picture vivid, Boolos needs to tease out the literal truth that lies behind the metaphor of the cumulative hierarchy being built up in stages. Work is needed. The metaphor tells us, "when one is told that a set is a collection into a whole of definite elements of our thought, one thinks: Here are some things. Now we bind them up into a whole. *Now* we have a set" (p. 18). But, speaking literally, at least if we are realists, we will agree that the sets have been there all along, independent of our constructive activity. Boolos gives axioms, strongly influenced by Dana Scott (JSL L 544), that give mathematical precision to the idea of sets built up in a cumulative hierarchy.

The effort succeeds in justifying the axioms other than choice and replacement, which, Boolos thinks, do not follow inevitably from the concept of a cumulative hierarchy. Indeed, Boolos has doubts about replacement, because he finds it hard to swallow that such enormous sets as the axiom demands exist. He thinks it is a definite question whether the replacement axioms are true or whether there are inaccessible cardinals, rejecting the view, originated by Zermelo (1255) and developed by Charles Parsons (JSL LIII 321(7)), that the quantifiers used in set theory are ambiguous, so that today's proper classes may become tomorrow's sets. It is a definite question whether the replacement axioms are true, but he thinks the answer may well be "No."

The work on second-order logic studies plural quantification, as found in Geach and Kaplan's example, "There are some critics who admire only one another." Boolos argues that one misrepresents the sentence if one takes it to be about sets or classes or fusions of critics. The sentence talks about critics, and it is to critics, and critics only, that the sentence is ontologically committed.

The sentence is important because, as Kaplan shows, it cannot be faithfully translated into a first-order idiom. Indeed, Boolos shows that the logic of plural quantification is intertranslatable with the version of full second-order logic that allows quantified variables to take the place of monadic predicates. One gets a categorical characterization of the natural numbers by adjoining to Robinson arithmetic the axiom, "It is not the case that there are some numbers among which none is least."

The crucial question for Boolos is not whether second-order quantification should be esteemed with the honorific "logic," but whether second-order methods are applicable even when the domain of discourse is unrestricted in extent. If we follow Quine's advice to regard second-order quantification as "set theory in sheep's clothing" (JSL LIII 303, p. 66), we shall be forbidden to employ it in contexts in which our domain includes all sets, on pain of Russell's paradox, whereas Boolos permits its use even when our domain includes everything there is.

The principal burden of the section on Frege, which makes up the second and largest part of the book, is to acquit Frege of the slander that his logical work is a failure. Frege's logical work is deeply flawed—everybody knows that—but it is surprising how little of the work is tainted by paradox.

The flaw is Basic Law (V), which postulates a one-one second-level function—that is, a function of functions—associating an object with each first-order function. That, at least, is the usual diagnosis of Russell's paradox, which Boolos endorses, rejecting, after careful analysis, Michael Dummett's suggestions (JSL LXI 1389, Chapter 17) that the blame for Russell's paradox should be placed either on Frege's use of impredicative second-order variables or on his failure to recognize that the Russell set for a given domain of discourse lies outside that domain of discourse.

Frege uses Basic Law (V) to derive Hume's principle that two concepts have the same number if and only if the things that fall under them can be brought into one-one correspondence. After this derivation, Basic Law (V) plays no further significant role. The task of the remainder of the ***Grundgesetze*** is to derive the laws of arithmetic from Hume's principle. Now Hume's principle is consistent (if second-order arithmetic is consistent), as John Burgess has shown (review of Crispin Wright's ***Frege's conception of numbers as objects***, in ***Philosophical review***, vol. 93 (1984), pp. 638–640), and the derivation of its arithmetical consequences is perfectly wholesome mathematics. Indeed, it is masterly mathematics. Boolos argues convincingly that Frege's derivation of the second-order Peano axioms from Hume's principle is one of the premier accomplishments of modern logic. Had it not been for Boolos's skillful efforts, Frege's splendid proof, sketched in the ***Grundlagen*** (though with a gap) and completed in the ***Grundgesetze***, would likely have been lost to modern readers, hidden in the intricacies of Frege's eccentric notation.

Hume's principle can be thought of as giving the meaning of the phrase "same number," in the same way that the principle (cited in the ***Grundlagen***, §64) that two lines have the same direction if and only if they are parallel can be regarded as giving the meaning of "same direction." Is the principle therefore analytic? Crispin Wright (JSL L 252) thinks so. He notes that, since Hume's principle is true on no finite domain, this entails that it is analytic that there are infinitely many individuals, and he is willing to accept that conclusion. Boolos disagrees. He notes that the logical form of Hume's principle—an object is associated with each concept in such a way that two concepts have the same associated object just in case they stand in a given equivalence relation—does not guarantee the principle's truth, or even its consistency, as Basic Law (V) illustrates dramatically. Moreover, it is possible to devise two consistent principles of that form that, taken together, are inconsistent. Can we at least be confident that Hume's principle is consistent? It is demonstrably equiconsistent with second-order arithmetic, but can we be incorruptibly certain that "some hotshot Russell of the 23rd Century won't do for us what Russell did for Frege?" (p. 313). Moreover, our best theory of infinite number—ZF—tells us that certain outsize concepts, such as *ordinal number*, lack an associated cardinal number. So if we regard Hume's principle as analytically true, we must regard ZF as analytically false, an unwelcome outcome.

The book concludes with lighter pieces. There are two papers on the *sorites* paradox, in which it is shown that, contrary to what one might have guessed, the formalization of the inference from "0 is small" and "Every successor of a small number is small" to "1,000,000,000 is small" takes only a few lines of natural deduction proof.

There is *Gödel's second incompleteness theorem explained in words of one syllable*, delivered to the Visiting Committee examining the MIT Department of Linguistics and Philosophy, to the committee's great delight.

There is a devilishly difficult variant on Raymond Smullyan's delightful "knights and knaves" puzzles.

Boolos asked an illuminating question about Cantor's result that there is no one-one function from the power set $\mathcal{P}(S)$ of a set S to S. The usual proof is indirect. One first shows that there is no surjection: $S \to \mathcal{P}(S)$ by showing how, given a function $g : S \to \mathcal{P}(S)$, one can define a subset of S that is not in the range of g. Next one shows that, if we had an injection: $\mathcal{P}(S) \to S$, one could use it to define a surjection: $S \to \mathcal{P}(S)$. Boolos asks, Can we obtain a direct proof that enables us to define, for a given function $f : \mathcal{P}(S) \to S$, an explicit counterexample to the thesis that f is one-one? The affirmative answer Boolos obtains discerns an unexpected close connection between Cantor's theorem and the Burali-Forti paradox.

Finally, there is a short paper based on the startling observation, by Boolos's undergraduate student Michael Ernst, that the English phase '‘*b*’ appended to ‘*a*’' is ambiguous. It might be taken to denote a quotation name, but it might instead be taken to denote a name of ‘*ab*’. To devise a variant of ordinary quotation that is free of ambiguity is a nontrivial problem that Boolos solved with glee. The paper illustrates a notable trait. Boolos was able to undertake serious philosophy and difficult mathematics, and he was able to make the entire enterprise enormous fun. It was a gift that will be sorely missed.

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DIRK VAN DALEN. ***Mystic, geometer, and intuitionist. The life of L. E. J. Brouwer.*** Volume 1. ***The dawning revolution.*** Clarendon Press, Oxford University Press, Oxford, New York, etc., 1999, xv + 440 pp.

The chapter headings are (1) *Child and student*, (2) *Mathematics and mysticism*, (3) *The dissertation*, (4) *Cantor–Schoenflies topology*, (5) *The new topology*, (6) *Making a career*, (7) *The war years*, (8) *Mathematics after the war*, (9) *Politics and mathematics*, (10) *The breakthrough*. A bibliography of all of Brouwer's writings and a list of contents of the second volume are included.

There exist a few biographies of leading logicians of the 20th century, most recently Dawson's on Gödel (***Logical dilemmas***, A K Peters, 1997), and Hodges's on Turing some seventeen years ago (JSL L 1065). A biography of Tarski is in the making and another one on Gentzen is about to appear. For one working in logic and foundations of mathematics, such biographies can satisfy a natural curiosity, but they can also broaden one's view.

The first volume of Dirk van Dalen's biography tells the story of the events in the life of Brouwer to about 1930. This is the second book dedicated to Brouwer's life and work, following Walter van Stigt's ***Brouwer's intuitionism*** (JSL LVI 1499). Van Dalen's book is decidedly more biographical, whereas Van Stigt tries to give systematic historically narrated introductions to Brouwer's philosophy, intuitionistic logic, and intuitionistic mathematics. (From the present book one learns the correct writing of Dutch names: lower-case van if the first name is included, otherwise upper-case Van.)

The "Grundlagenstreit," or debate about foundations of mathematics between Hilbert, Brouwer, and their disciples, was one of the great debates of the 20th century, comparable to the debate on foundations of quantum physics between Einstein and Bohr, or to the debate on foundations of probability. Most of the book can be read by anyone interested in intellectual history, in contrast to Van Stigt's book. It is to a large extent a story of persons, war, and politics. A window to young Brouwer's private mind is opened by the correspondence with the poet Carel Adama van Scheltema, lasting well over two decades from 1898 when Brouwer was merely seventeen. Five years later, Brouwer's confidence in his own capacities and appreciation of Scheltema's art made him compare the two friends to two kings each living in his own country (p. 37). The contrast to the recent Gödel biography is striking; one comes to know the person Brouwer from reading the Van Dalen book far more than one gets to know the person Kurt Gödel from reading the Dawson book

A chapter describes Brouwer's work in topology in the years around 1910. The general mathematical reader will know Brouwer's fixed point theorem, now set in a historical context of the development of topology. The topological works brought Brouwer international recognition and a position as professor in Amsterdam. Nothing supports the sometimes-heard claim that he first cunningly kept his intuitionism to himself, while securing a position through work in more "ordinary" mathematics. Rather, intuitionism was only germinating in Brouwer's mind at that time.

The book describes incidents testifying to a person who easily carries differences of opinion to absurd levels, with no respect for the position of those whom he has judged to be at fault. Denjoy, a Frenchman with a position in Holland, refused to referee a paper by a German in 1919. That was part of the international boycott against Germans after WW I. Brouwer demanded action by the academy against "this new mocking of our native country," but his protests brought no results. Still in 1922 we find him publishing at his own initiative and cost a pamphlet about the failure of the minister of education to respond to his earlier writings in the matter.

Against such negative events, one can put the contemporary testimony by Hermann Weyl (from a letter to Felix Klein): "Brouwer is a person I love with all my heart. I have now visited him in his home in Holland, and the simple, beautiful, pure life in which I took part there for a few days, completely and totally confirmed the impression I had made of him" (p. 298).

At that time, the stage for the fight about foundations of mathematics between Brouwer and Hilbert was set by Weyl's paper *Über die neue Grundlagenkrise der Mathematik* (192***4***) of 1921. The "Grundlagenstreit" Van Dalen sees as totally futile: one that "in principle could have been solved in a couple of workshops and personal meetings," as he writes on the very first page of his *Preface*. The views of some later workers along the lines of Hilbert's program, notably Gentzen in his 1938 review of the status of research into foundations of mathematics (JSL III 166), accord well with Van Dalen's statement. And if this is so, the debate about foundations is to an even greater extent a debate between persons. The full story will be told in the second volume where one chapter bears the same title as Van Dalen's much admired article, *The war of the frogs and the mice* (***The mathematical intelligencer***, vol. 12 (1990), pp. 17–31).

In describing the character of Brouwer's mature intuitionism, Van Dalen makes a strong contrast to the views of Kronecker and the French "semi-intuitionists." The essence of Brouwer's intuitionism is the theory of the continuum as based on choice sequences, and is not reducible to the arithmetic of natural numbers. Van Dalen (p. 316) puts his finger in a very precise place, the argument of 1918, for what is now known as Brouwer's *continuity principle*. The intuitionistic continuum is seen as a space of *choice sequences* of natural numbers, depicted as branches in a tree, starting with an empty root element, and branching denumerably at each node. Constructive functions defined over choice sequences must be computable from finite initial segments of the sequences. Should there be a point of discontinuity, the value would depend on a whole infinite choice sequence. Thus follows the continuity principle for constructive functions over the intuitionistic reals.

Van Dalen gives the credit for the continuity principle to Brouwer who "had conjectured the continuity theorem before anybody else" (p. 384). It must be pointed out, however, that an argument to that effect can be found in a 1912 paper by Émile Borel, *Le calcul des intégrales définies*: A real number is said to be calculable if one knows how to obtain arbitrarily good rational approximations to it. A real function is calculable if its value is calculable for a calculable argument, and therefore, "a function cannot be calculable unless it is continuous" (p. 835, all page references to Borel are to ***Œuvres de Émile Borel***, Paris 1972). The opposite to a calculable function is what Borel calls an *asymptotically definable* function, the values of which do not depend on any finite development of the argument. Here the bounds of Kroneckerian and semi-intuitionist reductionism are being passed.

Four years earlier, in 1908, Borel had given his solution to Richard's paradox (132***3***): Put in lexicographical order all "finite sequences of words that define a real number." Now "the least real number not defined by a finite number of words" diagonalizes the previous, denumerable set of reals. The essence of Borel's solution is that it is not *decidable* whether a sequence of words defines a real number, thus, there is no effective lexicographical order to diagonalize (ibid., p. 1273). One can suggest a definition that depends on the solution of an unsolved mathematical problem: "Among possible definitions, there are ones that suppose the solution of a mathematical problem." The example Borel gives is to switch 7 and 8 everywhere in the decimal expansion of π and to ask if the resulting number is algebraic (p. 1274). Here we have a predecessor to the idea of "Brouwerian counterexamples." In a paper published in 1909 (132***4***), Borel contemplates a continuum, the elements of which are determined by a "denumerable infinity of successive choices" (p. 1268), thereby anticipating Brouwer's notion of a choice sequence. (This has been pointed out by Jervell in a note of 1979, *From the axiom of choice to choice sequences*, published seventeen years later in the ***Nordic journal of philosophical logic***, vol. 1 (1996), pp. 95–98). In the 1909 paper, Borel points at the difference between a proposition and its double negation (p. 1269). By 1912 Borel saw that the equality of two calculable real numbers is in general undecidable (p. 831). There is reason to think that Brouwer had at least some knowledge of Borel's ideas; the 1909 paper was given in a 1908 conference where Brouwer also participated.

Brouwer is known for a forbidding terminology and notation in his intuitionistic papers from the twenties. They are, indeed, hard to read. In the late 1920's, Karl Menger went as far as devising a "dictionary" for translating Brouwer's language into what everyone conversant with set theory could understand. Van Dalen's biography contains extensive descriptions of a philosophical circle Brouwer took part in, the "signific circle." The circle aimed at a moral betterment of humankind through a socio-linguistic reform. Brouwer himself believed that old words contain moral connotations that can lead to evil thoughts. For him, language was in the first place a means for getting power over others. Thus, the strange and specifically intuitionistic vocabulary (and notation, in part still followed by some intuitionists) is part of a utopian program of language revision. It would be interesting to have some comparison to other utopian ideas about language at the time, say, attempts at creating one single international language, supposed to promote world peace and understanding between peoples.

The motivation for intuitionism was always a philosophical one. This is so for Brouwer himself, and Heyting shared the same opinion. Still in the late seventies, Michael Dummett in his ***Elements of intuitionism*** (JSL XLIV 276) maintained that intuitionism would be pointless without its philosophical motivation. This attitude is challenged by Bishop's constructive analysis and by much recent work on constructive systems motivated by connections to computer science: Constructivity supports computability, so if the premisses have a finitary computational meaning, the conclusion has it too. As Van Dalen describes it, the characteristically Brouwerian intuitionism begins with the extension of intuitionistic principles beyond arithmetic and it remains to be seen how much of the genuinely intuitionistic—in contrast to merely constructive—will be reappearing in work on computational logic.

Finally, we can hope for a more accurate proofreading for the second volume; there are numerous small misprints in the present one.

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MARTIN DAVIS. ***The universal computer. The road from Leibniz to Turing.*** W. W. Norton & Company, New York and London 2000, xii + 257 pp.

This book surveys developments in mathematics and logic that led the way to the design and construction of modern digital computers. In contrast to such works as Herman Goldstine's ***The computer from Pascal to von Neumann*** (Princeton University Press, 1972), its focus is on conceptual rather than technological advances. Thus it devotes only brief mention to such figures as Joseph-Marie Jacquard, Charles Babbage, Howard Aiken, John Atanasoff, John Mauchly, and J. Presper Eckert, Jr., and none at all to persons such as Herman Hollerith or Vannevar Bush. Its first seven chapters are devoted instead to biographical sketches of the lives and conceptual contributions of Gottfried Leibniz, George Boole, Gottlob Frege, Georg Cantor, David Hilbert, Kurt Gödel, and Alan Turing—those "whose ideas made it all possible," yet who are "easy to overlook" in the glare of the rapid advance of computer technology and "the truly remarkable accomplishments of the engineers" (p. xii). Machines themselves are discussed in the penultimate chapter, which focuses on von Neumann and the Moore School.

Among the topics discussed are Leibniz's vision of a logical calculus for manipulating the symbols of a universal language of thought; Boole's and Frege's mathematical treatments of propositional and predicate logic; Cantor's diagonal method and his theory of transfinite numbers; Hilbert's proof-theoretic program, and the fundamental limitations imposed on it by Gödel's incompleteness results; and finally, Turing's theoretical model of a universal computing machine and his demonstration of the unsolvability of its halting problem.

All those topics will be familiar to logicians, of course. But that is not the audience for whom Davis's account is intended. It is aimed rather at those who, lacking mathematical sophistication, most need to be made aware of the mathematicians and logicians whose ideas formed the theoretical foundation for the computer revolution.

It is not easy to write an account of such matters for a popular audience. Davis, however, has done a masterful job; his book is not only informative, but very enjoyable reading. The discussions of technical matters are clear, accurate, and accessible, and the biographical sketches bring their subjects to life, placing them in the historical and social contexts of their times and exposing their faults (such as Frege's anti-Semitism) as well their virtues. Nor does Davis refrain from social commentary: Turing's demise, for example, is described in a section entitled *A grateful nation rewards its hero*, and the volume concludes with an epilogue in which Davis stresses that, apart from Turing (and, one should add, von Neumann), none of the conceptual innovators portrayed in the book foresaw the application of their ideas to the construction of all-purpose digital computers—a fact that he believes "underscores ... the futility of predicting where [ideas] will lead" and shows that those who, like Leibniz's patrons, the Dukes of Hanover, "try to steer [scientists] in directions deemed most likely to provide quick results" are "likely [both] to [fail] in the short run [and to] shortchange ... the future" (p. 209).

The book contains few errors of any consequence. The most important is a misnumbering of endnotes in Chapter 6, for which the author has suggested the following correction: The endnote marker on page 129 for note 30 should be deleted, and each subsequent endnote marker in that chapter should be decreased by 1; then, in the endnotes themselves, the text of endnote 31 should be replaced by that of endnote 32, and endnote 32 should read "[Dawson], p. 147." In addition, on page 18, in the line between Proposition 5 and Proposition 6, "A for B" should be "C for A." Page 25, line 14, "women" should be "woman." Page 46, line 17, "Adolph" should be "Adolf." Page 128, line 7, "sanataria" should be "sanatoria." Page 195, line 6 from below, "desultary" should be "desultory." Finally, in the title of Gödel's Gibbs lecture, page 135, the word "*Philosophical*" should be inserted before "*Implications*" (an error reproduced from that in Volume III of Gödel's ***Collected works***, Oxford University Press, 1995).

This book should appeal to a wide audience. In particular, it is to be hoped that its message will be conveyed to those who wonder what logicians do and whether they are gainfully employed.

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Handbook of recursive mathematics, Volume 1, ***Recursive model theory***, edited by Yu. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel, with V. W. Marek, Studies in logic and the foundations of mathematics, vol. 138, Elsevier, Amsterdam etc. 1998, xlvi + 620 pp.—therein:

Y. L. ERSHOV, S. S. GONCHAROV, A. NERODE, and J. B. REMMEL. *Introduction to the Handbook of recursive mathematics*. Pp. vii–xlvi.

VALENTINA S. HARIZANOV. *Pure computable model theory*. Pp. 3–114.

YU. L. ERSHOV and S. S. GONCHAROV. *Elementary theories and their constructive models*. Pp. 115–165.

C. J. ASH. *Isomorphic recursive structures*. Pp. 167–181.

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S. S. GONCHAROV. *Autostable models and algorithmic dimensions*. Pp. 261–287.

J. F. KNIGHT. *Degrees of models*. Pp. 289–309.

A. S. MOROZOV. *Groups of computable automorphisms*. Pp. 311–345.

M. G. PERETYAT′KIN. *Constructive models of finitely axiomatizable theories*. Pp. 347–379.

D. CENZER and J. B. REMMEL. *Complexity theoretic model theory and algebra*. Pp. 381–513.

I. KALANTARI. *A bibliography of recursive algebra and recursive model theory*. Pp. 515–581.

V. BRATTKA and I. KALANTARI. *A bibliography of recursive analysis and recursive topology*. Pp. 583–620.

The first volume of this two-volume work consists of ten excellently written papers together with a comprehensive bibliography on computable model theory, computable algebra, and computable analysis and topology. The papers are written by leading experts in computable algebra, model theory, and computability theory. The volume is highly recommended to anyone interested in logic, computability, and complexity.

The volume begins with an introduction that gives an overview to effective mathematics by discussing its roots, Hilbert's program, developments of computable function theory and computable mathematics, modern history and some general themes of computable mathematics, relations with other subjects, and future developments.

Roughly, one can put the papers of the volume into several categories. The papers *Elementary theories and their constructive models* by Yu. Ershov and S. Goncharov, *Pure computable model theory* by V. Harizanov, and *Constructive models of finitely axiomatizable theories* by M. Peretyat′kin survey and discuss results related to the interactions between computability theory and model theory. An emphasis in these papers is given to understanding the effective content of results in model theory and the computability-theoretic complexity of certain (e.g. first-order) properties of models. The papers *Isomorphic recursive structures* by C. Ash, *Autostable models and algorithmic dimensions* by S. Goncharov, and *Degrees of models* by J. Knight discuss issues related to presentations of algebraic systems and their interactions, such as computable isomorphisms types, finding presentations that are least complex in a certain precise sense, and issues between computability-theoretic complexity and model-theoretic definability.

The paper *Groups of computable automorphisms* by A. Morozov deals with computable automorphisms of models and the relationship between structural properties of models and their computable automorphisms. The paper *Σ-definability of algebraic structures* by Yu. Ershov gives a general view on computable models by studying Σ-definable algebraic systems in admissible sets. Results and problems related to complexity-theoretic model theory and algebra are presented in the paper by D. Cenzer and J. Remmel, *Complexity theoretic model theory and algebra*.

Now I shall give a short overview of each paper in order of its appearance in the volume. The paper *Pure computable model theory* by V. Harizanov is an excellent survey of results in computable model theory. The paper supplies a model-theoretic background as well as proofs of many basic results in computable model theory. These include, for example, the effective completeness theorem, the effective omitting types theorem, and computability-theoretic conditions that are sufficient and necessary for a theory to have various types of decidable models including prime and saturated models. The paper revisits Vaught's theorem from a computability-theoretic point of view, and discusses decidable models of theories with countably many models (e.g. ω-categorical theories, Ehrenfeucht theories, $\aleph_1$-categorical theories). The paper also discusses the issue of the relationship between decidable models and indiscernibles, degrees of models, automorphisms, and computable models. In short, the paper is a well-written survey and is recommended to anyone who is interested in computable model theory. The paper can be used as a reference by exerts and as a basis for a seminar in logic.

The paper *Elementary theories and their constructive models* by Yu. Ershov and S. Goncharov is an excellent introduction to the language and concepts used in computable model theory by leading experts from the Novosibirsk school of logic. The paper consists of two parts. The first part introduces the concept of numbering, and the notions of constructive and strongly constructive models. This part of the paper provides elementary results and introduces the concept of computable families of constructive models. One of the highlights of the first part is a full proof of the theorem that shows that a homogeneous model of any complete theory T has a strong constructivization if and only if the space of all types realized in the model is computable and has an effective extension property. Several nontrivial and important corollaries of this result are then obtained. In the second part, the paper discusses decidable Boolean algebras. An emphasis is given to the study of strongly constructive Boolean algebras of a given characteristic. In particular, the second part provides several theorems related to proper constructive elementary extensions of constructive Boolean algebras to strongly constructive ones. For example, it is shown that any strongly constructive Boolean algebra has a proper strongly constructive elementary extension. In addition it is shown that any strongly constructive Boolean algebra has a constructive elementary extension to a strongly constructive saturated Boolean algebra. The paper is highly recommended to those interested in computable model theory and computable Boolean algebras.

The third paper *Isomorphic recursive structures* is written by C. Ash, who was one of the leading experts in the study of recursive structures. The paper discusses a range of issues related to recursive stability and categoricity, Δ^0_α-stability and Δ^0_α-categoricity, intrinsically Σ^0_α-relations, α-systems, and non-recursive structures. These are fundamental topics of study in the area. Due to the untimely death of the author, the final version of this paper was prepared by J. Knight.

V. Dobritsa contributes to this volume by presenting the paper *Computable classes of constructive models* and gives a research line full of natural and open questions and ideas related to uniformity issues on presentations of models or classes of models. An emphasis is given to studying the relationships between different uniform presentations of classes of computable models. A basic concept of the paper is the following. Assume we are given a class of computable models. In particular, the class can consist of certain computable copies of a given model. Then having two effective uniform enumerations, in a certain precise sense, of the class, one can ask how these two enumerations are related to each other. This relationship leads to natural partial orders on the set of all uniform enumerations of the class, and hence one can study structural properties of these partial orders. This is a serious area of research where many concepts of pure computability theory can be applied, studied, and developed.

Yu. Ershov's paper is titled *Σ-definability of algebraic structures*. The notion of Σ-definability is a natural generalization of the notion of constructivization. Using the theory of admissible sets, Ershov defines the notions of A-constructivization in which computable operations and predicates are replaced by Σ-definable sets. Several interesting results are then provided that underline the relationship between definability and effectiveness. The paper is very well written, compact, and opens up a new research area to study the interplay between model theory, the notions of α-recursion theory and admissible sets, and a natural generalization of the theory of computable models.

S. Goncharov presents the paper *Autostable models and algorithmic dimensions*. An important part of this paper is devoted to presenting the concepts of algebraic, program, and uniform reducibilities between computable presentations of models. These reducibilities form partial orders. The paper motivates the study of these partially ordered sets and poses problems related to studying their properties.

The next paper *Degrees of models* is written by J. Knight. The paper is devoted to the study of Turing degrees of isomorphic copies of structures. The first part of the paper presents basic properties of the degrees of models. The second part studies and presents some known results concerning the relationship between degrees of structures and their jumps. Parts 3 and 4 of the paper are devoted to the study of degrees of models of arithmetic and models of highly non-recursive theories. One of the important concepts presented in this paper concerns the relationships between Scott sets and degrees of models. The paper also presents some results that guarantee the existence of computable models for highly non-recursive theories.

The next paper *Groups of computable automorphisms* by A. Morozov is very well written and gives a comprehensive overview of results and techniques needed to study computable isomorphisms of models. Basically, the paper consists of two parts. The first part contains general results on computable automorphisms concerning complexity of orbits, complexity of groups of computable automorphisms, computability criterion of the groups of computable automorphisms, and so forth. The second part contains results concerning the interplay between computable automorphisms of Boolean algebras and algebraic properties of computable Boolean algebras. The paper is very useful for anyone interested in computable aspects of automorphisms and can be used as a basis for graduate seminars.

The paper *Constructive models of finitely axiomatizable theories* by M. Peretyat′kin surveys a number of results on finitely axiomatizable theories. The paper presents basic ideas and surveys of how one can convert a given theory into a finitely axiomatized theory, preserving model-theoretic and computability-theoretic properties of models of the original theory. The paper contains very nice results related to complexity of sets of formulas true in some natural classes of models. The author of the paper poses many interesting and important sets of problems whose solutions would advance the area considerably.

Finally, the last paper *Complexity theoretic model theory and algebra* by D. Cenzer and J. Remmel is devoted to complexity-theoretic model theory and algebra. The paper discusses complexity-theoretic issues of constructions and results obtained in computable model theory and algebra. Two main themes are concerned with existence results on the one hand, and properties of feasible (e.g. polynomial-time) structures on the other hand. The paper contains many results about polynomial-time Boolean algebras, linear orders, Abelian groups, and vector spaces, and discusses problems related to uniqueness of polynomial-time presentations. The area is new, promising, and possesses a potential for obtaining interesting and unexpected results that show the interplay between complexity theory, algebra, and model theory. It is a very well-written paper that can be used by those interested in the interactions between complexity, model theory, and algebra. The paper is a solid basis for a logic seminar.

There are several topics that could equally well be covered in the handbook. One of the natural important topics could be on computably enumerable algebras and model theory. Given the size of the handbook, however, inclusion of this and some other topics seemed to be an impossible task. The ***Handbook*** is surely an excellent source of ideas, concepts, topics, open problems, and results for any mathematician and computer scientist who is interested in effective mathematics, computations, complexity, structures, logic, and their interplay.

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Handbook of recursive mathematics, Volume 2, ***Recursive algebra, analysis and combinatorics***, edited by Yu. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel, with V. W. Marek, Studies in logic and the foundations of mathematics, vol. 139, Elsevier, Amsterdam etc. 1998, xlvi + pp. 621–1372—therein:

Y. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel. *Introduction to the Handbook of recursive mathematics*. A reprint of BSL VII 66. Pp. vii–xlvi.

D. Cenzer and J. B. Remmel. Π^0_1 *classes in mathematics*. Pp. 623–821.

R. G. Downey. *Computability theory and linear orderings*. Pp. 823–976.

R. G. Downey and J. B. Remmel. *Computable algebras and closure systems: coding properties*. Pp. 977–1039.

W. Gasarch. *A survey of recursive combinatorics*. Pp. 1041–1176.

N. G. Khisamiev. *Constructive abelian groups*. Pp. 1177–1231.

H. A. Kierstead. *Recursive and on-line graph coloring*. Pp. 1233–1269.

K. Ko. *Polynomial-time computability in analysis*. Pp. 1271–1317.

S. P. Odintsov. *Generally constructive Boolean algebras*. Pp. 1319–1354.

S. G. Simpson and J. Rao. *Reverse algebra*. Pp. 1355–1372.

Recursive mathematics is, very briefly, the study of how, in what ways, and to what extent the functions that exist in mathematics can be constrained to be recursive. The most obvious example is that of two dense, unbordered, linear orderings (like the rational numbers). The well-known process for computing the isomorphism is recursive (provided the orderings themselves possess certain "obvious" recursive properties). But there are many more areas where similar things can happen.

This 700+ page volume is the second half of a mammoth compilation of results in recursive mathematics, virtually all of which have been obtained within the last thirty years. The mathematics itself has its roots in work of Mal′cev in the East and the intuitionists in the West, in particular in the study of (classical) mathematical structures using recursive functions that was initiated by Ershov in the East and Dekker and Myhill in the West. Nerode's initiative at Cornell in the sixties and particularly the seventies, however, was the prime mover for the spectacular developments over three decades.

The first volume of this work (see the preceding review) looked at recursive mathematics in the context of mathematical logic. For the most part this second volume concentrates on specific structures mainly in discrete mathematics: in particular, linear orderings and graphs, Boolean algebra, combinatorics, and Abelian groups, and there is only a short section on analysis. (The reason for this brevity is the existence of M. Pour-El and I. Richards's book ***Computability in analysis and physics***, JSL LVI 749.) There is also a short article on the connection with the reverse mathematics of Simpson et al.

This review will now comment on the individual articles.

Π_1^0 *classes in mathematics* by D. Cenzer and J. B. Remmel: Early work of Kleene on his notation system for ordinals, O, drew attention to the branches in a recursively described tree. A Π_1^0 class is the set of infinite paths through a recursive tree. The authors first develop their properties and then give a number of examples of their application in constructing (first-order) logical theories, non-monotonic rule systems, Boolean algebras, graphs, partial orderings, linear orderings, and real functions and games, all with recursive (or r.e.) properties. Finally they show how to obtain feasible (that is, polynomial-time computable) solutions to many problems from recursive ones.

Computability theory and linear orderings by R. G. Downey: As noted at the beginning of this review, dense linear orderings probably represent the first place where a mathematician meets a memorable constructive proof. It is not surprising, therefore, that this area has been one of the (if not *the*) most developed part of recursive mathematics. Although one can find isolated results about the recursive mathematics of linear orderings even from before 1960 which Downey cites as "folklore," it was Rosenstein's book, ***Linear orderings***, JSL XLVIII 1207, that established the groundwork. Thereafter growth was rapid.

Downey mimics the development. He starts off very basically from Turing machines but takes the reader rapidly into priority arguments. (He uses the newspeak "computable" for "recursive" making it necessary to translate when looking at the original papers, but charmingly retains "r.e." as an abbreviation for "computably enumerable.")

First comes the question: What are the nice orderings? And how many recursive properties do we need? Then there is the question: What is the relation between the classical order type and the recursive one? Can we in fact have a recursive representative of an order type? The answer is obviously "No" from a straight countability argument, but that is only the beginning of the story. Next Downey looks at embedding properties and in particular at the question of whether we can introduce the word "computable" (*scilicet* recursive) everywhere and have the new version of the theorem still hold. Finally there is the question of extending partial orders to linear ones. He also opens up the question of studying not merely recursive linear orderings but ones that have their first-order theory decidable. (Cf. Odintsov's article below.) The article concludes with an impressive and useful bibliography.

Computable algebras and closure systems: coding properties by R. G. Downey and J. B. Remmel: Like linear orderings, vector spaces have long been accessible to effective procedures. Usually one is given a vector space basis from which to start a construction. Downey and Remmel consider the standard generalization of this idea, Steinitz systems, in various structures including vector spaces.

A survey of recursive combinatorics by W. Gasarch: This is a masterly and, despite its size of 136 pages, delightful survey. The strategy is made clear: first ask if the recursive analogue is true, which it usually is not; secondly can we modify the statement to make it true?—and the answer is often "Yes"; and finally, if we can do neither, can we classify the problem further up the arithmetic hierarchy? Standard examples are chosen: König's lemma, Ramsey's theorem, graph colorings, Hall's theorem (on bipartite graphs, i.e., graphs with male and female points with edges only between opposite sexes), Dilworth's theorem (partial orders of finite widths have covers of that width), together with brief hints for the treatment of still another dozen combinatorial problems.

Constructive abelian groups by N. G. Khisamiev: Ulm's theorem (and Ulm invariants) have long been known to categorize Abelian p-groups and, like linear orderings, might have been expected to attract a great deal of attention. In fact this has not happened and most of what has been done has been done in the East. Khisamiev's article presents the recursive version of this theorem and then goes on to consider other Abelian groups and their first-order theories.

Recursive and on-line graph coloring by H. A. Kierstead: "On-line colouring" means colouring the points in a graph one by one as you are given them. So here we see that recursive mathematics is approaching practical utility. Indeed Kierstead ends his article with an example of dynamic storage allocation (for example, in computer memory).

Polynomial-time computability in analysis by K. Ko: Despite the fact that everything about the real numbers looks far too classical and infinite, very good work has been done, with Harvey Friedman, Pour-El and Richards, and Ker-I Ko playing principal rôles. One regards a recursive real as a recursive approximation (by rationals) that has a recursive modulus of convergence. Then one can start talking about recursive or computable real functions. From this basis one can go on to ask about maxima of functions and integrating functions and even do things with differential equations. One can even talk about polynomial-time real functions. Ko tantalizes us with the basic results. This is an area that seems difficult and rewarding. It is certainly pointing out problems and opportunities.

Generally constructive Boolean algebras by S. P. Odintsov: The authors in this second volume have, for the most part, primarily concerned themselves with structures that have their basic relations and operators recursive (groups, linear orderings, etc.); the articles in the first volume principally deal with questions about general structures (models of a theory, etc.). On the other hand Odintsov looks at both aspects. That is to say that, while he restricts himself to Boolean algebras, he also looks at the amount of constructivity in their theories. This means, for example, in looking at the complexity of the satisfaction relation for formulae or specific classes of formulae, e.g. the formulae with $\leqq n$ alternations of quantifiers, one may find all of them to be decidable. If this is the case then one can construct nice Boolean algebras with constructive properties. As usual, however, there are limits to what can be achieved.

Reverse algebra by S. G. Simpson and J. Rao: This odd little article seems out of place in the book and does little to tie the two subjects of recursive mathematics and reverse mathematics together.

The lists of references in some of the articles were sometimes prepared before the papers had appeared so details are incomplete in some instances but otherwise the book is very well produced and excellently structured. The book could have been shorter if proofs had been omitted more but in some cases there are a number of new results here. The reviewer's main regret is that this book is not available on line. Trying to keep track of the multitude of definitions or to find exactly the result one wants or needs is difficult (except in Gasarch's article)!

In summation, however, the compilation is not only a useful resource but also a large stepping stone on the way to getting mathematicians to think about how their structures might be used in a practical way on computers.

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P. G. Odifreddi. ***Classical recursion theory.*** Volume II. Studies in logic and the foundations of mathematics, vol. 143. Elsevier, Amsterdam etc. 1999, xvi + 949 pp.

After reviewing Volume I of this series for the Journal in 1990 (JSL LV 1307), I was waiting eagerly and increasingly impatiently through the 1990's for the appearance of Volume II. By 1998 I had nearly given up hope—projects that are delayed for nearly ten years have a high probability of never being completed. Now that I have seen, handled, and prowled about in the book, however, I am astounded that it took *only* ten years to complete. The child described in that review has become both a master gardener and a professional geologist.

Notice that I do not claim to have *read* the book; actually reading (and digesting) the contents of this book would take (me, at least) something between $\aleph_0$ and $\aleph_1$ years. I have been a recursion-theorist for some thirty-five years and have a knowledge of the topics covered in this book ranging from nodding acquaintance to active research, but for each one of these topics the presentation here gave me new insight into the richness and intricacy of the subject. Furthermore, the range of these topics is so large that even a superficial treatment would be a monumental task. This book is an achievement of the highest order and will be an essential resident on the shelf of any logician, aspiring or mature.

Let me now illustrate these general remarks with some details. The first two chapters (VII and VIII, continuing the numbering of Volume I) consist of 356 dense pages of information on the complexity of effective computation. We move from abstract complexity measures to abstract time and space measures to nearly every complexity class known to man, from REAL (time) to EXPSPACE^n to the Grzegorczyk classes to the γ-recursive functions for γ in the far reaches of the Veblen–Bachmann hierarchy. Rates of growth, time and space trade-offs, domination relations, and much more are studied for all of these—more than a full year's course in complexity theory could think of covering.

Chapters IX and X contain 283 pages devoted to the recursively enumerable (r.e.) sets and degrees. Volume I gave us the basic facts about these topics including a priority-free solution to Post's problem. Here in Chapter IX we are immersed in the web of interconnections of (r-)maximal, (hyper)hypersimple, filters of the lattice $\mathcal{E}$ of r.e. sets, and high$_n$ and low$_n$ degrees. There are a few results about automorphisms of $\mathcal{E}$, but many classic results, such as the fact that the maximal sets form an orbit, are only mentioned in passing. The last two sections of Chapter IX make contact with the material on complexity via speedability and inductive inference for r.e. sets.

The first four sections of Chapter X take us through finite and infinite injury priority arguments. After the Friedberg–Mučnik theorem and the thickness lemma, the tree method is introduced and used to give alternative proofs. The axiomatic demands of priority arguments are evaluated in the contexts of admissible sets and bounded arithmetic. Section X.5 studies in considerable detail the structure of the many-one degrees of r.e. sets, and the remaining sections do the same for the Turing degrees, and in less detail for the 1-, tt-, wtt-, and btt-degrees. The focus is on establishing local results that lead to establishing the complexity and distinctness of the first-order theories of these structures. Among the missing in this chapter are $\mathbf{0}'''$ priority arguments (briefly mentioned) and the jump operator (treated in Chapter XI).

Chapter XI deals with the degrees below $\mathbf{0}'$. Here is a full treatment of jump- and hop-inversion, full-approximation methods, 1-generic degrees, cupping and capping, and minimal degrees and covers. Section XI.5 covers definability, automorphisms, and the first-order theory of the Turing degrees below $\mathbf{0}'$ and its relationship with other segments of Turing degrees. Section XI.6 considers briefly the many-one degrees of sets recursive in $\mathbf{0}'$.

Chapter XII is a relatively short (44 pages) analysis of the arithmetical sets, concentrating on forcing and its applications and Turing degrees of arithmetical sets. Chapter XIII deals with the arithmetical degrees—those generated by the relation "arithmetical in"—alone and with the natural corresponding jump operator, the ω-jump. This chapter contains some of the most complex arguments of the book, including Harrington's solution to the arithmetical analogue of Post's problem and M. Simpson's construction of a minimal pair of arithmetical degrees. The final Chapter XIV is on enumeration degrees.

For many books of mathematics, such a brief catalog of the contents would serve adequately as a review, but in this case it fails by far to do justice to the work. For one thing, without taking up considerably more space than I am allotted here, there is no way to convey the thoroughness with which most of these topics are treated. This book is a rarity in the field in being truly *scholarly*. With a few exceptions, some noted above, threads are followed from proved theorems to exercises with elaborate hints to long lists of references. Indeed, many of the exercises are probably for most readers solvable only by a trip to the library. A 21-page index and a 59-page bibliography provide superb resources for tracking down information. In most cases, the complete current state of knowledge is either presented or referenced. Of course everyone, myself included, will find favorite topics missing, but that is inevitable and should not detract from an appreciation of the high degree of completeness.

Also characteristic is the style of proofs throughout. Of necessity, given the scope of the book, many proofs are sketched. In some cases this seems to be a conscious style of presenting the underlying idea of a proof without getting (too) distracted by the details. I am not suggesting here that there is any lack of precision; when full details are given, they are impeccable. There are, however, many instances in which the directions are laid out in words and the reader is left to fill in the symbols.

A very useful feature is the attention paid to motivating the steps of the proof. For a (nearly random) example, consider the proof of Theorem XI.3.9 (p. 683), the capping theorem for the degrees below $\mathbf{0}'$. Near the beginning we read "The obvious attack ... is .... The obvious trouble with this is .... The idea is ... "; and later "If we insist in using the simple approach described above ... , then we run into trouble as follows. ... To avoid this, it is enough ... ." This is a common pattern; the author describes the natural strategy, then the obstructions that prevent this from working as expected, and finally the modifications that overcome the obstructions. Although following such an argument often makes greater demands on the reader than would the polished final version, I agree with what I assume is the author's view that it generally results in a deeper understanding of the reasoning.

As a final note, I want to comment on the ways that this book will be used. As for Volume I, the author admits that his twin goals of providing both an "adequate textbook, and a reference manual" are "somewhat irreconcilable." This seems even more true for Volume II than for Volume I; although the discussion above makes clear my admiration for the book in its reference function, the instructor using it as a text will need to give a large amount of guidance to keep students from being overwhelmed and distracted from the main themes of the course. That said, the book is suitable for use in a very wide variety of courses and will provide ample material for independent study for students planning eventual research in recursion theory.

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Richard A. Shore and Theodore A. Slaman. *Defining the Turing jump*. ***Mathematical research letters***, vol. 6 (1999), pp. 711–722.

The (Turing) jump is the operation on Turing degrees induced by the relativized halting problem. Thus for a (Turing) degree $\mathbf{d}$, the jump of $\mathbf{d}$, denoted $\mathbf{d}'$, is defined to be the degree of $\{e : \varphi_e^A(e) \text{ is defined}\}$, where $A \subseteq \omega$ is any set of degree $\mathbf{d}$, and φ_e^A is the eth computable partial function relative to the oracle A. The jump operation embodies the diagonal method of producing a degree strictly greater than a given degree, although its definition is exterior to the ordering of the degrees. In the paper under review, it is shown that the jump operator is first-order definable (without parameters) in the structure $(\mathcal{D}, \leqq)$ where $\mathcal{D}$ is the set of all degrees and $\leqq$ is its usual partial ordering. This is a remarkable result which gives a strong positive answer to the question raised by S. Kleene and E. Post in ***Annals of mathematics***, ser. 2 vol. 59 (1954), pp. 379–407 (JSL XXI 407), whether the jump operation (which was originally defined in that paper) is intrinsic to the ordering of degrees. The proof has two ingredients: (a) a theorem relating the jump and join operations on degrees (Theorem 2.1 of the paper under review), and (b) the definability of the double jump operation in $(\mathcal{D}, \leqq)$, which this paper states will be proved in a forthcoming monograph by Slaman and H. Woodin, ***Definability in degree structures***.

The study of definability in degree structures, and in the lattice of r.e. sets, has long been a central topic in computability theory. For the Turing degrees, it was shown by the reviewer and R. Shore in ***The journal of symbolic logic*** vol. 49 (1984), pp. 1205–1236, that the set of degrees of arithmetical sets is first-order definable in $(\mathcal{D}, \leqq)$. This proof used a join theorem for ω-r.e. operators to give a natural structural definition for the degrees of the arithmetical sets. In the same paper, it was pointed out that this result could be combined with earlier work of Shore to get a wide class of definability results for $(\mathcal{D}, \leqq)$, for example the definability of the singleton $\{\mathbf{0}^{\omega}\}$. These further results depend, however, on coding models of arithmetic into the degrees and the corresponding definitions are not "natural" in the sense that the definition of the arithmetical degrees is.

The status of the result that the jump is definable in $(\mathcal{D}, \leqq)$ is rather complicated. This result was originally announced by S. B. Cooper in ***Bulletin of the American Mathematical Society***, n.s. vol. 23 (1990), pp. 151–158. Cooper's argument used the "pseudojump" machinery of the reviewer and R. Shore (see the reference above) as well as a "non-splitting" theorem to be proved by a $\mathbf{0}'''$-priority construction reminiscent of A. H. Lachlan's non-splitting theorem (***Annals of mathematical logic***, vol. 9 (1976), pp. 307–365). The "pseudojump" part of the proof was clear, but the "non-splitting" part has not yet appeared in detail. Nonetheless, this result was generally accepted among computability theorists. The paper under review cites a surprising result by Shore and Slaman (to appear in *A splitting theorem for n-REA degrees*, ***Proceedings of the American Mathematical Society***) that refutes Cooper's "non-splitting" theorem. Subsequently, Cooper announced on his website www.amsta.leeds.ac.uk/~pmt6sbc that a modification of his original method would establish the definability of the jump. In light of the fact that Cooper's proof of this modified version and the Slaman–Woodin monograph mentioned above are both in preparation, no complete proof of the definability of the jump exists in print at the time of writing of this review (August, 2000).

The path to definability of the jump given in the paper under review uses a remarkable "join" theorem. The prototype for this theorem is due to D. Posner and R. Robinson (***The journal of symbolic logic***, vol. 46 (1981), pages 714–722): For any degree $\mathbf{a} > \mathbf{0}$ there is a degree $\mathbf{g}$ such that $\mathbf{g}' = \mathbf{g} \vee \mathbf{a}$, where $\mathbf{g} \vee \mathbf{a}$ denotes the least upper bound of the degrees $\mathbf{g}$ and $\mathbf{a}$. The natural extension of this result to finite iterations of the jump was raised as an open question in the paper of the reviewer and Shore mentioned above: For any integer $n > 0$ and any degree $\mathbf{a} \nleqq \mathbf{0}^{(n-1)}$, is there a degree $\mathbf{g}$ such that $\mathbf{g}^{(n)} = \mathbf{g} \vee \mathbf{a}$? Of course, the case $n = 1$ is just the Posner–Robinson join theorem, which is a relatively simple result. Even the case $n = 2$, however, was highly resistant over a period of years. The general case is proved as Theorem 2.1 of the paper under review. The case $n = 2$ of this theorem implies that $\mathbf{0}'$ is the greatest degree $\boldsymbol{z}$ such that there is no degree $\mathbf{g}$ with $\mathbf{g}'' = \mathbf{z} \vee \mathbf{g}$. Thus, $\mathbf{0}'$ is defined in terms of the double jump operation and the ordering of degrees (Theorem 1.14), and the jump operation is defined analogously by relativization (Theorem 1.15).

The proof of Theorem 2.1 is an elegant application of Kumabe–Slaman forcing, which was invented by Kumabe and Slaman to prove the analogue of Theorem 2.1 for $n = \omega$. The generic objects for this forcing are Turing functionals (which are essentially reduction procedures, but need not be computable). Thus, $\mathbf{g}$ of the theorem is the degree of a Turing functional Φ, and this functional is chosen so that $\Phi(A)$ has degree $\mathbf{g}^{(n)}$. The authors use Theorem 2.1 to prove a generalized version for n-REA operators as defined in the paper of the reviewer and Shore cited above and even extend it to α-REA operators for arbitrary computable ordinals α in Theorem 2.15.

As the authors point out, their path to defining the jump operator does not lead to a definition that is simple or natural in the obvious sense. The definition of the jump in terms of the double jump is certainly natural, as indicated above. But the definition of the double jump in the Slaman–Woodin monograph involves coding of models of arithmetic and other metamathematical considerations and is neither simple nor natural. For an interesting discussion of natural and unnatural definitions see Shore's paper, *Natural definability in degree structures*, ***Computability theory and its applications***, Contemporary mathematics, vol. 257, American Mathematical Society, Providence 2000, pp. 255–271.

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ULRICH KOHLENBACH. *On the arithmetical content of restricted forms of comprehension, choice and general uniform boundedness*. ***Annals of pure and applied logic***, vol. 95 (1998), pp. 257–285.

The paper under review can be considered as part of the larger project of *extracting additional information from proofs*. The project originated with the famous question of G. Kreisel: "What can be learned, if we know a proof of a theorem, compared to the mere fact of its truth?" Possible answers to this question usually take the form of extracting effective bounds from (ineffective) proofs, highlights being, for example, the pioneering result of G. Kreisel on Littlewood's theorem, *On the interpretation of non-finitist proofs* (JSL XVIII 78), and H. Luckhardt's analysis of Roth's theorem, *Herbrand-analysen zweier Beweise des Satzes von Roth: polynomiale Anzahlschranken* (***The journal of symbolic logic***, vol. 54 (1989), pp. 234–263). More recently, U. Kohlenbach used this methodology to extract effective bounds in Chebycheff approximation theory (see *Analysing proofs in analysis*, ***Logic: from foundations to applications***, Clarendon Press, Oxford University Press, 1966, pp. 225–260). See also S. Feferman, *Kreisel's "unwinding" program* (***Kreiseliana: about and around Georg Kreisel***, A K Peters, 1996, pp. 247–273), for a critical overall assessment of this line of research.

Technically, analysis of proofs often has the form of a reduction of an *expressively strong* second-order or finite-type system S to a *weak* first-order or quantifier-free system of arithmetic T, such as PRA or its fragments. The requirements are that (1) the strong system naturally formalizes the mathematical proof under consideration, whereas (2) the weak system allows for the extraction of reasonably effective bounds.

In proof theory one is traditionally interested in the reductions that yield conservation results. These concerns are, in particular, motivated by modern versions of the Hilbert program. For example, the conservativity of a fragment of analysis S over PRA can be interpreted as grounding the part of mathematics formalizable in S on purely finitistic means. Notice that a reduction of this type is the more informative, the stronger the system S (and the weaker the system T). This draws attention, in particular, to problems of the following type: *Determine possibly stronger natural systems S that are conservative over* PRA. (This problem does not have a unique solution: there are natural examples of systems S_0 and S_1, even in the language of first-order arithmetic, either of which is conservative over PRA such that $S_0 + S_1$ proves the totality of the Ackermann function. Thus, in principle, incomparable systems can be used to extract effective bounds from proofs.)

In the project of extracting computational information from proofs one is driven by different concerns. In practice, to simplify a proof analysis, one often deals with more general kinds of reductions, which preserve the rate of growth bounds, but need not yield conservation results. (For example, one often simplifies the proof analysis by freely using arbitrary true universal lemmas, or more complicated *analytical principles*, in the terminology of U. Kohlenbach, which do not contribute to the rate of growth.)

The paper under review represents a certain compromise between these two types of concerns. On the one hand, one is interested in the traditional type of conservation results between fragments of analysis in all finite types and fragments of arithmetic with regard to the classes of the prenex arithmetical hierarchy. On the other hand, one tries to isolate the fragments S that are closer to the systems occurring in actual "unwinding" of mathematical proofs.

One of the main results of the paper is an improvement of a well-known result of J. Paris (*A hierarchy of cuts in models of arithmetic*, JSL LI 1062) and H. Friedman (unpublished) on Π_{k+2} conservativity of the Π_k-collection principle over the Σ_k-induction schema in fragments of PA. Corollary 4.8 of the paper states that over some basic system of analysis in all finite types corresponding to the nth class of the Grzegorczyk hierarchy (denoted $E\text{-}G_nA^\omega$ in the paper) the following combination of schemata is conservative for Π^0_{k+2}-sentences over the Σ^0_k-induction principle $\Sigma^0_k\text{-}IA^-$ (where $^-$ denotes the absence of function parameters; type-0 number parameters are allowed; this system is a conservative extension of first-order $I\Sigma_k$): (0) $AC^{1,0}\text{-}qf$ (quantifier-free choice in types $\{1,0\}$); (1) $\Delta^0_{k+1}\text{-CA}^-$ (comprehension); (2) $\Pi^0_k\text{-AC}^-$ (choice); (3) *WKL* (weak König's lemma).

Since $\Pi^0_k\text{-AC}^-$ contains arithmetical Π_k-collection, this result improves the cited theorem of J. Paris and H. Friedman. Corollary 4.11 also shows that the above combination of schemata is, in fact, Π^0_{k+3}-conservative over the Π^0_k-collection principle without function parameters. This means that we deal here with an essentially second-order improvement of the Paris–Friedman theorem. The system given by (0)–(3) for $k = 1$ seems to be one of the strongest currently known natural conservative extensions of PRA.

This theory is natural and the result is interesting, because the schema $\Pi^0_k\text{-AC}^-$ often appears in formalizing mathematical arguments. For example, $\Pi^0_1\text{-AC}^-$ proves (a weak schematic version of) the principle of convergence of bounded monotone sequences of reals, whereas $\Pi^0_2\text{-AC}^-$ is needed to prove a suitable version of the existence of a limit superior for bounded sequences (a proof of this fact appears in the later publication by Kohlenbach, *Things that can and things that cannot be done in PRA*, ***Annals of pure and applied logic***, vol. 102 (2000), pp. 223–245).

At this point the reader may well ask: Why do we need to consider the rather awkward restrictions of the schemata to the function-parameter-free ones (and some other special families of formulas and systems that the reader encounters in the paper)? The answer is that from the point of view of the "practical" task of extracting effective bounds, only those principles are of significance that contribute as little as possible to the rate of growth of provable functions. Unrestricted use of function parameters is well known to be bad in this respect. Thus, the more common schemata, although they allow sometimes for more compact formulations, are of little use for the purposes of the project of extracting numerical bounds from proofs, as well as for the reductionistic aims of grounding much of mathematics on PRA.

Methods of the paper are based on two special techniques: *monotone functional interpretation* and the elimination of Skolem functions for a class of so-called *monotone formulas*, for which the main conservation results are proved. This class of formulas is not restricted in any of the usual classes Σ_n or Π_n, which allows one to state some results in greater generality. Another advantageous feature of the techniques is that they not only allow one to provide a characterization of provably recursive functions, but also of the provable type-2 functionals of a system. Details of the techniques, however, can be found only in earlier works of the author (*Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals*, ***Archive for mathematical logic***, vol. 36 (1996), pp. 31–71, and *Elimination of Skolem functions for monotone formulas in analysis*, ibid., vol. 37 (1998), pp. 363–390).

The paper is difficult to read for some objective and some subjective reasons. So, it may require some patience on the reader's part. The objective reason is that one deals here with a great variety of schemata and principles, which are often not so nicely formulated. The absence of sufficiently attractive and memorable *abstract* formulations is a general malaise of this subdiscipline, for which nobody has found a good cure so far. In a sense, this is the price one has to pay for the compromise between the traditional proof-theoretic concerns and the needs of practical proof-unwinding.

A subjective reason is that the author has chosen the style of the presentation, where the more general (but difficult to formulate) results have the status of "theorems" and "propositions," whereas some of the interesting and more memorable formulations are downgraded to "corollaries."

Despite these criticisms, I believe that the direction of research is motivated by the vital needs of proof theory, especially the search for new applications, which are especially important at present. The author is doing a very good and useful job on it. The results obtained so far are interesting and have a potential for further development of the subject. The techniques used can also be of value for the more traditional areas of proof theory.

Finally, I would like to make a small bibliographical correction to the claims made on page 260: J. Paris published his conservation result in *A hierarchy of cuts in models of arithmetic* (JSL LI 1062), not in his paper with L. Kirby, Σ_n*-collection schemas in arithmetic*, ***Logic Colloquium '77***, North-Holland, 1978, pp. 199–209.

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TOSHIYASU ARAI. *Some results on cut-elimination, provable well-orderings, induction and reflection*. ***Annals of pure and applied logic***, vol. 95 (1998), pp. 93–184.

The fact that this paper was originally titled *From the attic* is strong evidence that Arai's attic is more interesting than most. The paper is a collection of results gathered over the course of a decade or so, spanning a wide range of topics in proof theory, and tying up a number of loose ends. There are some new results here, but the general emphasis is on providing strengthenings and new proofs of previous results, many of them well known or folklore. Arai is always on the mark: while the paper does not break dramatically new ground, it is full of clever tricks, keen insights, satisfying observations, and nontrivial refinements, presented in a clean and elegant way.

For the most part, each of the eight sections can be read independently. Arai is to be commended for providing detailed references, contextual notes, and explanatory remarks. In the following, therefore, I will provide only a brief synopsis, glossing over many of the details and omitting citations that can be found in the paper itself.

Section 1 addresses the topic of provable well-orderings. Thanks to Gentzen, we know that any elementary recursive ordering that is provably well ordered in Peano arithmetic has order-type less than ε_0. Takeuti and, independently, Harrington have shown that if R is an elementary relation that is provably well ordered in PA, there is, moreover, a $<\varepsilon_0$-recursive comparison map between R and standard notation systems for ε_0. Arai uses a clever coding trick to provide two nice strengthenings: one can, in fact, find a comparison map that is *elementary*, and even under the weaker assumption that R is simply provably *well founded* (not necessarily totally ordered). This analysis carries over to reasonable extensions of PA, but Arai notes that it is open whether one can prove the same result for fragments.

In Section 2, Arai shows that one can extract elementary bounds on the increase in length when eliminating cuts from proofs in propositional sequent calculi. This result, which stands in sharp contrast to predicate logic, is well known, and the standard proofs are not difficult; but Arai uses instead a simple counting argument that highlights the difference between the propositional and predicate cases. He also uses these methods to refine some of the results in Beckmann's dissertation, involving separations of theories of bounded arithmetic with a predicate variable.

Section 3 strengthens some classic results due to Howard and Kreisel on the relationship between bar induction, comprehension, and reflection. A complementary result due to

H. Friedman on the relationship between bar induction and a principle of ω-model reflection can be found in Stephen Simpson's book, ***Subsystems of second order arithmetic*** (JSL LXIV 1356).

Section 4 solves a problem posed by Friedman on the equational calculus PRE, i.e., primitive recursive arithmetic minus the induction axioms. If f is a function symbol of PRE, let $Cl(f)$ denote f together with all the functions symbols used in its definition. If e is an equation and E is a set of equations, $Cl(e)$ and $Cl(E)$ are defined analogously to represent the set of function symbols found implicitly in e and E. A proof of e from E in PRE is said to be *direct* if every function symbol used in the proof occurs in $Cl(E) \cup Cl(e)$. Friedman gave two examples of provable entailments in PRE that cannot be established by using direct proofs: if S denotes the successor function, $x = y$ follows from $Sx = Sy$ by use of the predecessor function, and $y = z$ follows from $0 = Sx$ by using the conditional function. Arai's Theorem 4.1 provides the satisfying conclusion that these are, in a sense, the *only* counterexamples: if one adds these to the system as rules, every provable entailment has a direct proof. Arai uses this to show that the question whether an equation is consistent with PRE is decidable, but he notes that the decidability of provability for open formulas is still unresolved.

Buchholz has used a realizability interpretation to show that a certain intuitionistic fixed-point theory, $\widehat{ID}^{i}_{1}$(strong), is conservative over Heyting arithmetic for almost negative formulas. Section 5 provides an elegant use of Goodman's theorem to extend the conservation result to arbitrary arithmetic formulas.

It is a scandal of proof theory that various definitions of a theory's proof-theoretic ordinal need not coincide, though empirically it seems that for "natural" theories they do. Jäger and Primo have come across one of the few real counterexamples, with a second-order theory of weak-fixed-point definitions, whose Π^1_1 ordinal is that of Peano arithmetic, i.e., ε_0 (which amounts to saying that ε_0 is the least upper bound to the theory's provable well-orderings), but whose Π^0_2 ordinal is that of Σ^1_1-AC, i.e., $\varphi_{\varepsilon_0}(0)$ (which amounts to saying that the theory's provably total recursive functions are exactly the ones that are $< \varphi_{\varepsilon_0}(0)$-recursive). While the second fact can be obtained with a direct interpretation, Arai shows that the first fact follows easily from an observation due to Kreisel that adding arbitrary true Σ^1_1 sentences to a theory does not change its Π^1_1 ordinal. This proof is simpler than the one provided by Jäger and Primo, and yields a neat diagnosis of the rogue phenomenon.

In the late 1960's, Kreisel and Levy demonstrated a general relationship between reflection principles and induction, and in the late 1970's, Schmerl extended the analysis to transfinitely iterated reflection principles and transfinite induction. In Section 7, Arai develops some refinements, obtaining tight connections between transfinitely iterated reflection principles and theories with finitely many applications of a transfinite induction rule. (Related and partially overlapping results have been obtained independently by Beklemishev.)

Interest in the proof-theory community in the problem of bounding the lengths of derivations in various rewrite systems began with work by Dieter Hofbauer (see *Termination proofs by multiset path orderings imply primitive recursive derivation lengths*, ***Theoretical computer science***, vol. 105 (1992), pages 129–140). In the last section of this paper, Arai uses the results of Section 7 to obtain tight bounds on the length of derivations in rewrite systems compatible with a lexicographic path ordering, refining results due to Buchholz and Weiermann. Along the way, he also draws out connections between such rewrite systems, fast-growing hierarchies, slow-growing hierarchies, and a hierarchy of fragments of arithmetic between $I\Sigma_1$ and $I\Sigma_2$.

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James E. Baumgartner, Alan Taylor, and Stanley Wagon. *Ideals on uncountable cardinals*. ***Logic Colloquium '77, Proceedings of the colloquium held in Wrocław, August 1977***, edited by Angus Macintyre, Leszek Pacholski, and Jeff Paris, Studies in logic and the foundations of mathematics, vol. 96, North-Holland Publishing Company, Amsterdam, New York, and Oxford, 1978, pp. 67–77.

J. E. Baumgartner, A. D. Taylor, and S. Wagon. ***Structural properties of ideals.*** Dissertationes mathematicae (Rozprawy matematyczne), no. 197, Polska Akademia Nauk, Instytut Matematyczny, Warsaw 1982, 95 pp.

James E. Baumgartner and Alan D. Taylor. *Saturation properties of ideals in generic extensions*. ***Transactions of the American Mathematical Society***, vol. 270 (1982), pp. 557–574, and vol. 271 (1982), pp. 587–609.

The late sixties were the golden age of combinatorial set theory. Cohen's method of forcing provided a new powerful technique that made possible a detailed investigation of uncountable sets. Ultrafilters on ω and their structural properties received much attention, as did partition properties. At the same time, introduction of model-theoretic techniques into the study of large cardinals, by Keisler, Scott, Silver, and Solovay among others, led to a great number of new results on κ-complete ultrafilters and ideals on large cardinals.

The papers under review, by Baumgartner and his students, pursue a structural analysis of κ-complete ideals on an uncountable cardinal κ. The purpose of this detailed analysis was to initiate a systematic investigation of properties of such ideals that on the one hand are analogous to properties that had been studied in conjunction with ultrafilters on ω, and on the other hand generalize the structural theory of ultrafilters on measurable cardinals.

The first of the papers under review appeared in the proceedings of the European summer meeting of the Association for Symbolic Logic held in Wrocław in August 1977. It contains a list of results and an outline of the more substantial second work ***Structural properties of ideals*** that represents a detailed systematic study of κ-complete ideals on an uncountable cardinal κ. The work follows several major themes from recent literature on the subject and attempts a unifying approach. Among the topics and properties of ideals discussed at length are the Rudin–Keisler ordering, almost-disjoint families, saturation of ideals, generic ultrapowers, games, precipitous ideals, stationary sets, P-points, Q-points, selective ideals, partition properties, the Boolean algebra $P(\kappa)/I$, as well as many variations on these themes. The paper gives a list of many known results and proofs of many new ones. The *Appendix* gives a brief overview of the results and open problems.

The two-part paper by Baumgartner and Taylor deals exclusively with those properties of ideals that are related to saturation. In particular, precipitous and κ^+-saturated ideals are studied, and a new property, "presaturation," is introduced. The properties are investigated in the context of generic extensions and in relation to several other conjectures that arise in combinatorial set theory and in the theory of large cardinals.

Among the large number of concepts introduced or studied in these papers are some that have been forgotten or have fallen into obscurity in the past twenty years. But some of these concepts have become central to current research in set theory. Not coincidentally, the analysis of saturation of the ideal of non-stationary subsets of ω_1 has played an important role in the study of forcing axioms, large cardinals, and determinacy.

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C. Stephen Layman. ***The power of logic.*** Mayfield Publishing Company, Mountain View, Calif., London, and Toronto, 1999, ix + 566 pp.

This book is a clear and accessible, if not enormously innovative, presentation of the staples covered in a traditional introductory logic text. It includes treatments of arguments, syllogisms, fallacies, the rudiments of symbolic logic, induction, and probability. The book emphasizes diagramming argument structure more than most books do, has a novel and illuminating discussion of the role of metaphor in arguments from analogy, and ends with an unexpected chapter on sentential modal logic. But for the most part the topics, and the treatments of them, are standard.

The presentation of probability, for example, covers much of the same ground, in roughly the same way, as Skyrms's account in ***Choice and chance*** (JSL XLI 547). The treatment of sentential and predicate logic, which occupies three core chapters, also contain few novelties. The natural deduction system for the former comprises an unwieldy twenty rules, though, like most things in the book, they are presented in a clear and direct way. The chapter on informal fallacies is a speedy tour of fourteen fallacies in twenty-six pages, and many readers will feel that if this territory is worth covering at all (something some of us would doubt), it deserves more detail. Even more readers will wonder about the value of a chapter on the arcane art of syllogistic. Finally, one may question the book's inclusion of topics like appeals to pity and to force, while ignoring the last thirty years of research in behavioral decision theory, which tells us a good deal about the types of reasoning errors to which people are *actually* prone—and something about how to avoid them.

But these flaws, if flaws they be, are endemic to the genre, and if someone is in the market for a clear, well-organized text in the traditional mold, this book is better than many. It always gets to the point quickly and clearly, it is never silly or marred by lapses (like talk about a distinction between deductive and inductive arguments) of the sort one finds in more texts than one might imagine, and it contains a wealth of exercises drawn from the real world. The market is glutted, however, and a book like this needs some sort of edge to justify its adoption when there is so much to chose from. That edge is provided by the interactive tutorial for the book which resides on the internet at http://www.poweroflogic.com/cgi-pol/menu.cgi.

The book's website, which students using other introductory texts should also find helpful, begins with the expected true-false questions, multiple choice problems, and matching exercises. These pedestrian exercises become more useful than usual when run on a computer, because they force students out of their passivity and they supply instant feedback, including diagnostic messages and advice, when answers are wrong. The tutorial also contains a number of more innovative modules, including nicely interactive ones where users symbolize English sentences, construct truth tables, and devise counterexamples to invalid argument forms. But it really comes into its own with modules that allow students to diagram argument structure, use Venn diagrams, and construct proofs.

The software does not match the sophistication of the gee-whiz, *Star wars*-like programs of Barwise and Etchemendy, which for the foreseeable future will be in a league of their own. But unlike Barwise and Etchemendy's programs, the present software is accessible to anyone on the web. Besides, it covers somewhat different ground, and it covers it better than anything else I have seen.

The module on truth tables guides users through the construction of full and abbreviated tables and allows use of both to test arguments for validity. The module on the unduly-neglected topic of diagramming argument structure is innovative, though its basically text-based approach means that it lacks the immediate visual impact of diagrams, with arrows linking premisses and conclusions, drawn on the board. This is one place where we would expect more use of computer graphics in the future. The module on Venn diagrams, by contrast, makes effective use of graphics, though it is tightly yoked to syllogisms, and opportunities to use such diagrams to explain basic probabilistic concepts or to introduce basic ideas about diagrammatic reasoning are largely ignored. Since the book does not include these topics, the tutorial cannot be faulted for such omissions, of course, and in any case it could easily be extended to deal with such matters.

Finally, the proof-checker is a very sophisticated and useful tool. Although it does not completely eliminate the tedious feel of sitting in front of a typewriter, it is one of the few pieces of software that allows students to enter proofs in a relatively effortless way, and they can ask it, at any point in their proof, whether what they have entered thus far is correct. Since the proof checker recognizes so many natural deduction rules, students working with more restricted sets of rules could also use it with profit. In short, students who work through the exercises in this tutorial while reading the accompanying text will learn a good deal of logic in a reasonably painless way.

Software this good is much harder to write than most people suppose until they actually sit down to try. It needs to be done by people who combine logical expertise, experience teaching beginning logic, and programming skills. The present site was constructed by two philosophers who fit the bill: Colin Allen, who has written various innovative pieces of logic software, including *The logic daemon* proof checker, and Christopher Menzel, who has written extensively on the logic of properties but also put in time at Texas A&M's Knowledge Based Systems Laboratory. In the future we would hope for software with greater graphical capabilities, but this is a promising start. It provides just the ingredient that transforms a good textbook into one that is a cut above most of the competition.

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