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COMPLETELY ADDITIVE LIFTINGS

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The purpose of this communication is to survey a theory of liftings, as developed in author's thesis ([8]). The first result in this area was Shelah's construction of a model of set theory in which every automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, where Fin is the ideal of finite sets, is *trivial*, or in other words, it is induced by a function mapping integers into integers ([33]). (It is a classical result of W. Rudin [31] that under the Continuum Hypothesis there are automorphisms other than trivial ones.) Soon afterwards, Velickovic ([47]), was able to extract from Shelah's argument the fact that every automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ with a Baire-measurable lifting has to be trivial. This, for instance, implies that in Solovay's model ([36]) all automorphisms are trivial. Later on, an axiomatic approach was adopted and Shelah's conclusion was drawn first from the Proper Forcing Axiom (PFA) ([34]) and then from the milder Open Coloring Axiom (OCA) and Martin's Axiom (MA) ([48], see §5 for definitions). Both shifts from the quotient $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ to quotients over more general ideals $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and from automorphisms to arbitrary homomorphisms were made by Just in a series of papers ([14]–[17]), motivated by some problems in algebra ([7, pp. 38–39], [43, I.12.11], [45, Q48]) and topology ([46, p. 537]).

Building on this work, we study homomorphisms of analytic quotients, both simply definable and arbitrary ones, confining ourselves to rather large class of *analytic P-ideals*, i.e., those ideals which are analytic subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$ (after a natural identification of sets with their characteristic functions) and which are σ-directed under the ordering of inclusion mod finite. (See [35] for more information about these ideals.) It turns out that under OCA and MA the arbitrary homomorphisms can be reduced to the simply definable ones (see Theorem 5.2). Our main contribution is an isolation of a class of *nonpathological ideals*, sufficiently rich in the sense that it includes practically any example of analytic P-ideal that has appeared in the literature, and proving that homomorphisms between their quotients always have completely additive (see paragraph before Lemma 1.1) liftings.

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This result also reduces the complexity of the "embeddability of quotients" partial preordering, by showing that it coincides with the much simpler finite-to-one reduction of ideals, the so-called *Rudin-Blass* preordering. It also points to some similarities with the well-studied Borel-reducibility of equivalence relations induced by Borel ideals on the integers (see [11]).

Another unexpected consequence is that two nonpathological ideals give rise to isomorphic quotients only when they themselves are isomorphic (this answers an often asked question, see [7, pp. 38–39], [43, I.12.11], [45, Q48], [21], for this class of ideals). Finally, we get back to where it all started—every automorphism of a quotient over a nonpathological ideal is trivial.

The only non-P-ideal for which we prove analogous results is the ideal $\mathrm{Fin} \times \emptyset$ on the underlying set $\mathbb{N} \times \mathbb{N}$ generated by its vertical sections $\{n\} \times \mathbb{N}$. This result, however, appears to be very useful in topology. Namely, there has been a considerable interest in determining which spaces are continuous images of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, the Čech-Stone remainder of $\mathbb{N}$ (see [30], [46], [14], [4]), in particular among the "simply definable" spaces, such as $(\mathbb{N}^*)^2$ or $(\omega^2)^*$ (ω^2 is given its ordinal topology). Each one of these spaces is a Stone space of a quotient of some Borel subalgebra of $\mathcal{P}(\mathbb{N})$ over a countably generated ideal on $\mathbb{N}$. Therefore the question of which spaces are continuous images of $\mathbb{N}^*$ reduces to the question which of these algebras embed into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. We prove that, under OCA and MA, this preordering on algebras reduces to its version where it is required that these embeddings have continuous liftings. This enables us to give a rather complete answer to this question. These results are also used to describe all automorphisms of any finite power of $\mathbb{N}^*$.

In §1 we introduce the basic notions. In §2 we analyze homomorphisms with Baire-measurable liftings, and show that the question whether such liftings can always be turned into completely additive ones is equivalent to a finite combinatorial problem, which is answered in §3. In §4 we discuss some applications of completely additive liftings. Arbitrary homomorphisms are studied in §5, by using the additional axioms OCA and MA. In §6 we study gaps in some analytic quotients, and in §7 our results about liftings are applied to topology. The results of §6 and §7 have been obtained after the completion of [8], and they will appear in [9]. In §8 we mention some open problems. We have outlined selected proofs to give a flavor of this subject. An interested reader is invited to look for more details in [8] and [9].

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§1. Basic definitions. Our objects of study are ideals on $\mathbb{N}$ (or some other countable set) of the form $\mathcal{I} = \{ A : \lim_n \varphi(A \setminus [1, n)) = 0 \}$ where φ is a *lower semicontinuous submeasure* on $\mathbb{N}$, that is it satisfies $\varphi(\emptyset) = 0$,

$\varphi(A)+\varphi(B)\geq\varphi(A\cup B)$, and $\varphi(A)=\lim_{n\to\infty}\varphi(A\cap[1,n])$ for all A and B. This class of ideals is quite large, as it coincides with the class of analytic P-ideals on $\mathbb{N}$ (see [35]). Some of the classical examples of such ideals are:

1. Fin, the ideal of all finite subsets of $\mathbb{N}$.
2. *Summable ideals*,

$$\mathcal{I}_f=\left\{A\,:\,\sum_{n\in A}f(n)<\infty\right\}$$

for some $f:\mathbb{N}\to(0,\infty)$.

3. *Density zero ideal*,

$$\mathcal{Z}_0=\left\{A\,:\,\limsup_n\frac{|A\cap[1,n]|}{n}=0\right\}.$$

4. *Logarithmic density zero ideal*,

$$\mathcal{Z}_{\log}=\left\{A\,:\,\limsup_n\frac{1}{\log n}\sum_{i\in A\cap[1,n]}\frac{1}{i}=0\right\}.$$

5. $\emptyset\times\mathrm{Fin}=\{\,A\subseteq\mathbb{N}\times\mathbb{N}\,:\,A\cap\{n\}\times\mathbb{N}$ is finite for all $n\,\}$.

(The underlying set of the ideal $\emptyset\times\mathrm{Fin}$ is $\mathbb{N}\times\mathbb{N}$ instead of $\mathbb{N}$, but it is understood that this is merely a technical convenience.) These simply looking ideals, however, give rise to quotient Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ which can be quite complex, for example, quotients over F_σ ideals (like Fin or summable ideals) are always countably saturated. The basic question considered in this paper is the following: How does a change of the ideal $\mathcal{I}$ effect the change of its quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$? Various instances of this question were asked in the literature ([7, pp. 38–39], [43, I.12.11], [45, Q48], [21]). First of all, the question has to be put in a proper setting since, for example, the Continuum Hypothesis implies that all quotients over F_σ-ideals are saturated and therefore isomorphic ([18]). We take two, apparently different, approaches to overcome this problem. Understanding a connecting map (such as homomorphism) $\Phi\colon\mathcal{P}(\mathbb{N})/\mathcal{I}\to\mathcal{P}(\mathbb{N})/\mathcal{J}$ between quotients requires understanding a *lifting*, namely a map $\Phi_*\colon\mathcal{P}(\mathbb{N})\to\mathcal{P}(\mathbb{N})$ which induces Φ by making the diagram

$$\begin{tikzcd}
\mathcal{P}(\mathbb{N}) \arrow[r, "\Phi_*"] \arrow[d, "\pi_{\mathcal{I}}"'] & \mathcal{P}(\mathbb{N}) \arrow[d, "\pi_{\mathcal{J}}"] \\
\mathcal{P}(\mathbb{N})/\mathcal{I} \arrow[r, "\Phi"] & \mathcal{P}(\mathbb{N})/\mathcal{J}
\end{tikzcd}$$

commute. The first approach is to consider only those homomorphisms with liftings which are simple either topologically (like continuous or Baire-measurable) or algebraically, like *additive* (ones which themselves are homomorphisms of $\mathcal{P}(\mathbb{N})$) or *completely additive*, shortly c. a., (ones which moreover respect infinitary operations, namely such that $\Phi_*(A)=\bigcup_{n\in A}\Phi_*(\{n\})$

for all A). Considering Baire-measurable liftings is sufficiently general, because by a famous result of Solovay ([36]) every function that can be defined using reals and ordinals as parameters is likely to be Baire-measurable. By the following well-known result we can make a further reduction.

LEMMA 1.1. *If a homomorphism* $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ *has a Baire-measurable lifting then it also has a continuous lifting.*

PROOF. Let $\Phi_*\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a Baire-measurable lifting of Φ. Since Φ_* is continuous on a dense G_δ set $\bigcap_n U_n$, we can recursively choose sequences $1 = n_1 < n_2 < \dots$ and $t_i \subseteq [n_i, n_{i+1})$ $(i \in \mathbb{N})$ such that $X \cap [n_i, n_{i+1}) = t_i$ implies $X \in \bigcap_{j=1}^{i} U_j$. If $A_0 = \bigcup_i [n_{2i}, n_{2i+1})$, $C_0 = \bigcup_i t_{2i}$, $A_1 = \bigcup_i [n_{2i+1}, n_{2i+2})$, and $C_1 = \bigcup_i t_{2i+1}$, then

$$\Theta(B) = (\Phi_*((B \cap A_0) \cup C_1) \cap \Phi_*(A_0)) \cup (\Phi_*((B \cap A_1) \cup C_0) \cap \Phi_*(A_1))$$

is a continuous lifting of Φ. $\dashv$

We have reproduced this simple proof to display a reoccurring theme in this subject—lifting Φ_* is transformed into another one with stronger properties by finding sequences $\{n_i\}$ and $t_i \subseteq [n_i, n_{i+1})$ such that Φ_* is in some sense "stabilized" on the basic open set $\{X : X \cap [n_i, n_{i+1}) = t_i\}$. These so-called *stabilizers* are subsequently used to transform Φ_* into a more canonical lifting of the same homomorphism.

Every c. a. lifting Φ_* is continuous, for it is determined by the sequence $\Phi_*(\{n\})$ $(n \in \mathbb{N})$. In [39], Todorcevic posed the following, rather strong conjecture about liftings:

CONJECTURE 1.2 (STRONG LIFTING CONJECTURE, SLC). *Every Baire-measurable lifting of a homomorphism between quotients over analytic P-ideals can be turned into a c. a. one.*

Since c. a. maps behave like some sort of integrals, it is not unreasonable to say that $\mathcal{I}$ has the *Radon-Nikodym property* if every Baire-measurable lifting of a homomorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ can be turned into a c. a. one. So SLC can be reformulated as a statement that all analytic P-ideals have the Radon-Nikodym property. To see how relevant SLC is to our basic question, let us introduce some preorderings on the ideals.

(O1) (*Rudin-Blass preordering*) $\mathcal{I} \leq_{RB} \mathcal{J}$ if there is a finite-to-one $h\colon \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{I}$ if and only if $h^{-1}(A) \in \mathcal{J}$.

(O2) (*Baire embeddability*) $\mathcal{I} \leq_{BE} \mathcal{J}$ if there is a monomorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ with a Baire-measurable lifting.

It is clear that $\mathcal{I} \leq_{RB} \mathcal{J}$ implies $\mathcal{I} \leq_{BE} \mathcal{J}$, and that the converse is true whenever $\mathcal{J}$ has the Radon-Nikodym property. In the case of isomorphisms, the Radon-Nikodym property has even stronger consequences. Two quotients are *Baire-isomorphic* if there is an isomorphism which has a Baire-measurable lifting. Ideals $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic*, $\mathcal{I} \approx \mathcal{J}$, if and only if there is a bijection $\pi\colon A \to B$ between sets A, B of integers such that $\mathbb{N} \setminus B \in \mathcal{J}$ and $C \in \mathcal{I}$

if and only if $\pi''C \in \mathcal{J}$. Note that in most cases π can be a permutation of the integers.

PROPOSITION 1.3. *If ideals $\mathcal{I}$ and $\mathcal{J}$ have the Radon-Nikodym property, then the quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are Baire-isomorphic if and only if $\mathcal{I} \approx \mathcal{J}$.*

PROOF. Let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be an isomorphism and let h_1 and h_2 be such that Φ_{h_1} and Φ_{h_2} are liftings of Φ and Φ^{-1} respectively. If $\pi = h_1 \circ h_2$ is a bijection on a set whose complement is in $\mathcal{I}$, then this is the required isomorphism. So let us consider the set $C = \{n : \pi(n) \neq n\}$. By a well-known fact ([3, Lemma 9.1]) we can find a partition $C = C_0 \cup C_1 \cup C_2$ such that $C_i \cap \pi''C_i = \emptyset$ for $i = 0, 1, 2$. This clearly implies that these three sets, and therefore C as well, are in $\mathcal{I}$. ⊣

§2. Homomorphisms with Baire-measurable liftings. In this section we prove a weaker, yet quite useful (see proof of Theorem 5.2), version of SLC.

DEFINITION 2.1. If $\{n_i\}$ is a strictly increasing sequence of integers, $\{v_i\}$ is a sequence of disjoint finite sets of integers, and H_i is a function mapping subsets of the interval $[n_i, n_{i+1})$ to subsets of v_i, then define a mapping $\Psi_H : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$\Psi_H(X) = \bigcup_{i=1}^{\infty} H_i(X \cap [n_i, n_{i+1})).$$

C. a. liftings are a special case of Ψ_H, when $n_i = i$ and $H_i(\emptyset) = \emptyset$ for all i. The following is proved by using stabilizers (see also [15]).

THEOREM 2.2. *Every Baire-measurable lifting of a homomorphism between quotients over analytic P-ideals can be turned into one of the form Ψ_H.*

COROLLARY 2.3. *The ideal* Fin *has the Radon-Nikodym property.*

PROOF. If infinitely many of H_is are not additive, infinitely many of them fail to be additive in the same way. If, for example, there are $s_i, t_i \subseteq [n_i, n_{i+1})$ such that $H_i(s_i \cup t_i) \neq H_i(s_i) \cup H_i(t_i)$ for infinitely many i, then for $S = \bigcup_i s_i$ and $T = \bigcup_i t_i$ we have $\Psi_H(S) \cup \Psi_H(T) \neq^* \Psi_H(S \cup T)$, a contradiction. Therefore all but finitely many of H_i are additive, and Ψ_H is equal (modulo finite) to a c. a. function, which is the desired lifting. ⊣

This simple fact gives a hint that, in order to resolve SLC, we have to analyze maps H_i and turn them into additive maps without changing Ψ_H modulo the ideal $\mathcal{I}$. Let us take a look at the case when $\mathcal{I}$ is summable. Fix a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}_f$ and define a measure μ_f on $\mathbb{N}$ by $\mu_f(A) = \sum_{n\in A} f(n)$ so that

$$\mathcal{I}_f = \operatorname{Exh}(\mu_f) = \{ A : \lim_n \mu_f(A \setminus [1, n]) = 0 \}.$$

If Ψ_H is a lifting of Φ guaranteed by Theorem 2.2, let ε_i be the "error" of H_i, namely the maximum of measures of sets

$$(H_i(s) \cup H_i(t)) \,\Delta\, H_i(s \cup t) \quad \text{and} \quad H_i(s^{\complement}) \,\Delta\, H_i(s)^{\complement}$$

for $s, t \subseteq [n_i, n_{i+1})$. We say that H_i is an *ε_i-approximate homomorphism* with respect to μ. An argument analogous to one in the proof of Corollary 2.3 shows that $\sum_i \varepsilon_i < \infty$. Assume for a moment that for some fixed $K < \infty$ and all i there is a homomorphism $G_i\colon \mathcal{P}([n_i, n_{i+1}) \to \mathcal{P}(v_i)$ such that

$$\max_s \mu(H_i(s) \,\Delta\, G_i(s)) \leq K \cdot \varepsilon.$$

if this holds we say H_i is *$K \cdot \varepsilon$-approximated* by G_i. C. a. mapping Ψ_G obtained from Ψ_H by replacing H_i with G_i satisfies (let $X_i = X \cap [n_i, n_{i+1})$)

$$\mu_f(\Psi_H(X) \,\Delta\, \Psi_G(X)) \leq \mu_f\left(\bigcup_{i=1}^{\infty}(H_i(X_i) \,\Delta\, G_i(X_i))\right) \leq \sum_{i=1}^{\infty} K \cdot \varepsilon_i < \infty$$

for all X, and therefore it is a c. a. lifting of Φ. This uniform bound K turns out to be exactly what is needed, namely:

Proposition 2.4. *The following are equivalent*:

1. *all summable ideals have the Radon-Nikodym property.*
2. *there is a constant $K_M < \infty$ such that every ε-approximate homomorphism with respect to some measure can be $K_M \cdot \varepsilon$-approximated by a homomorphism.*

An analysis of liftings of the form Ψ_H in the case of arbitrary analytic P-ideals leads to an analogous result.

Proposition 2.5. *The following are equivalent*:

3. *all analytic P-ideals have the Radon-Nikodym property.*
4. *there is a constant $K < \infty$ such that every ε-approximate homomorphism with respect to some submeasure can be $K \cdot \varepsilon$-approximated by a homomorphism.*

§3. Approximate homomorphisms.

The last two propositions show that SLC is equivalent to a problem in finitary combinatorics, and the following is our basic result about this reformulation.

Theorem 3.1. *If $H : \{0,1\}^m \to \{0,1\}^n$ is an ε-approximate homomorphism with respect to some measure μ supported by $\{1, \dots, n\}$, then it can be $1007 \cdot \varepsilon$-approximated by a homomorphism.*

Let us outline the idea of the proof. For $j \leq n$ let

$$A_j = \{\, s \in \{0,1\}^m \;:\; H(s) \ni j \,\}.$$

Fix a small enough $\delta > 0$. Since H is an ε-approximate homomorphism, Fubini's theorem for the product $\{0,1\}^m \times \{1, \dots, n\}$ implies that the set (ν

is a uniform probability measure supported by $\{0,1\}^m$)

$$\{\, j \leq n \,:\, \nu\{\, s \,:\, s, s^{\complement} \in A_j \text{ or } s, s^{\complement} \notin A_j \,\} > \delta \,\}$$

has measure $< \varepsilon/\delta$. In other words, exactly one of s and $s^{\complement}$ is in A_j for almost all s and almost all j. Similarly, almost all A_j satisfy $s \cap t \in A_j$ if and only if $s, t \in A_j$ and $s \cup t \in A_j$ if and only if either s or t is in A_j, so we say that A_j "δ-almost everywhere" looks like an ultrafilter for all j. A key fact in the proof is:

LEMMA 3.2. *If $A_j \subseteq \{0,1\}^m$ looks like an ultrafilter δ-almost everywhere, then there is $k = k(j) \leq m$ such that $\nu(A \,\Delta\, \langle k\rangle) < 42\delta$.*

By neglecting a subset of $\{1,\dots,n\}$ of small measure, we can assume this applies to all A_j. Then it can be proved that the homomorphism $\Phi\colon \{0,1\}^m \to \{0,1\}^n$ defined by $\Phi(s) = \{\, j \,:\, k(j) \in s \,\}$ is a required approximation to H.

By a result of Christensen ([2]), a submeasure satisfies Fubini's theorem if and only if it is equal to the supremum of all measures μ such that $\mu(A) \leq \varphi(A)$ for all A. We call these *nonpathological submeasures* (this is not exactly the standard terminology—see, e.g., [19]). By Christensen's theorem, we have a following version of Theorem 3.1.

THEOREM 3.3. *If $H : \{0,1\}^m \to \{0,1\}^n$ is an ε-approximate homomorphism with respect to some nonpathological submeasure φ supported by $\{1,\dots,n\}$, then it can be $1007 \cdot \varepsilon$-approximated by a homomorphism.*

We can not relax the assumptions further by dropping the requirement that φ be nonpathological. Namely, for every $M < \infty$ there is a 1-approximate homomorphism with respect to some (pathological) submeasure φ which can not be $M \cdot \varepsilon$-approximated by a homomorphism (see [10]). Pathological submeasures are well-studied objects, especially in connection with the Control Measure Problem (see [19], [24]).

Similar stability problems for ε-approximately additive functions $H : \mathbb{R} \to \mathbb{R}$ and set-functions $H : \{0,1\}^m \to \mathbb{R}$ analogous to (2) of Proposition 2.4 have been previously considered (see [20]), and all these problems belong to a general "stability program" initiated long ago by S. Ulam (see [43, §VI.1], [44, §V.4]). We have recently learned from Professor A. Urquhart that objects related to ε-approximate homomorphisms have been used in Computer Science in the study of circuit complexity (see [1]), and we plan to examine the possible connection.

§4. Radon-Nikodym property. By results of the previous two sections, SLC is false in general. However, naturally occurring submeasures tend to be nonpathological, and the class of ideals induced by nonpathological submeasures, or *nonpathological ideals*, is sufficiently rich to include essentially all examples of analytic P-ideals occurring in the literature, for example the ideals 1–5 from §1 are all nonpathological. Therefore we have the following.

THEOREM 4.1. *All nonpathological ideals, in particular* $\emptyset \times \mathrm{Fin}$ *and all summable and density ideals, have the Radon-Nikodym property.*

Therefore in a substantial class of ideals Baire-embeddability reduces to the much simpler finite-to-one reduction, so that the basic question has a very plausible answer. We also have the following, more precise, version of Proposition 1.3.

PROPOSITION 4.2. *If* $\mathcal{I}$ *is nonpathological, then the quotients* $\mathcal{P}(\mathbb{N})/\mathcal{J}$ *and* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *are Baire-isomorphic if and only if* $\mathcal{I} \approx \mathcal{J}$.

PROOF. Note that we do not require that $\mathcal{J}$ is nonpathological, or even that it is analytic. However, $\mathcal{J} \leq_{BE} \mathcal{I}$ implies that $\mathcal{J}$ is an analytic P-ideal (by Theorem 8 of [39]), and it moreover implies that $\mathcal{J}$ is nonpathological: If $\varphi_{\mathcal{I}}$ is a nonpathological submeasure such that $\mathcal{I} = \mathrm{Exh}(\varphi_{\mathcal{I}})$ then $\varphi_{\mathcal{J}}(s) = \varphi_{\mathcal{I}}(\Phi_*(s))$ (Φ_* is a c. a. lifting) is nonpathological and $\mathcal{J} = \mathrm{Exh}(\varphi_{\mathcal{J}})$. Therefore the conclusion follows by Proposition 1.3. ⊣

This enables us to give some answers to a "definable" version of the basic question by performing some straightforward calculations. The question whether quotients over ideals $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are isomorphic was asked by P. Erdős and S. Ulam long ago (see [7, p. 38–39], [43, I.12.11], [45, Q48]), and solved by Just-Krawczyk ([18]) and Just ([15], [17]), relatively recently.

COROLLARY 4.3. *Quotients over the ideals* Fin, $\emptyset \times \mathrm{Fin}\, \mathcal{I}_{1/n}$, $\mathcal{I}_{1/\sqrt{n}}$, $\mathcal{Z}_0$ *and* $\mathcal{Z}_{\log}$ *are pairwise not Baire-isomorphic.*

Theorem 4.1 also reduces the $\mathbf{\Sigma}^1_2$ relation of Baire-embeddability to the simpler Rudin-Blass reduction between ideals. In the case of summable ideals this relation is even $\mathbf{\Sigma}^0_2$ (Lemma 7.5 of [8]), and this enables us to embed $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into the structure of summable ideals under the $\leq_{BE}$ (or $\leq_{RB}$) preordering, showing that this is a complicated structure (compare with [25] and [29]). A picture of some analytic ideals under the $\leq_{BE}$ preordering obtained using the above results, [35], and routine computations looks as follows (see [8, §6]).

$$\begin{tikzcd}
\text{non-P-ideals} & \mathcal{Z}_0,\ \mathcal{Z}_{\log} & \mathcal{I}_{1/\sqrt{n}} & \\
\mathrm{Fin}\times\emptyset \arrow[u] & \emptyset\times\mathrm{Fin} \arrow[u] & \mathcal{I}_{1/n} \arrow[u] & (X) \\
& \mathrm{Fin} \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are, although nonisomorphic, embeddable into one another. It is an open problem whether there are any analytic ideals in the class (X), i.e., ones which are not $\leq_{BE}$-comparable to either of the ideals $\mathrm{Fin} \times \emptyset$, $\emptyset \times \mathrm{Fin}$ or any summable ideal. This diagram and problem strikingly resemble the situation with the Borel-cardinality of quotients (see [21]). Yet these two preorderings do differ—for example, there are many Borel subalgebras of $\mathcal{P}(\mathbb{N} \times \mathbb{N})/\operatorname{Fin} \times \emptyset$ strictly between $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and $\mathcal{P}(\mathbb{N} \times \mathbb{N})/\operatorname{Fin} \times \emptyset$ in the Baire-embeddability sense, while this interval is empty in the case of Borel-cardinality preordering (see [22]).

§5. **Arbitrary homomorphisms under OCA.** An approach alternative to the study of homomorphisms with topologically simple liftings is to consider arbitrary liftings, assuming some alternative to CH. It turns out that the right alternative is a natural version of a two-dimensional Perfect Set Property known under the name of Open Coloring Axiom (OCA) introduced by S. Todorcevic ([40]). For a set X let $[X]^2$ be the family of all unordered pairs of its elements, $\{\{x, y\} : x \neq y \text{ and } x, y \in X\}$. A partition $[X]^2 = K_0 \cup K_1$ is *open* if K_0, when identified with a symmetric subset of $X \times X$, is open in the product topology. We say that a subset Y of X is K_i-*homogeneous* if $[Y]^2$ is included in K_i $(i = 0, 1)$.

(OCA) If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is an open partition, then either X has an uncountable K_0-homogeneous subset or it can be covered by a countable family of K_1-homogeneous sets.

We say that a set covered by countably many K_1-homogeneous sets is σ-K_1-*homogeneous*. We often say that a partition K is open if there is a separable metric refinement of the usual topology on X which makes K open. OCA was first shown to be relevant in this context by Velickovic ([48]) and Just ([17]). Recall that the first result in this area was Shelah's celebrated consistency proof of the statement saying that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ have c. a. liftings ([33]). The situation with arbitrary homomorphisms rather than automorphisms is by far more complex: If there is a maximal nonprincipal ideal $\mathcal{I}$ on $\mathbb{N}$, then there exists a monomorphism between two analytic quotients with no Baire-measurable, and therefore with no c. a., lifting. It can be constructed by splitting $\mathbb{N}$ into two disjoint infinite sets, A_1 and A_2, and then amalgamating the embedding $\Phi_1 \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(A_1)/\operatorname{Fin}$ with some monomorphism $\Phi_2 \colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(A_2)/\operatorname{Fin}$ as follows

$$
\begin{tikzcd}
& \mathcal{P}(A_1)/\operatorname{Fin} \arrow[dr, "\mathrm{id}"] & \\
\mathcal{P}(\mathbb{N})/\operatorname{Fin} \arrow[ur, "\Phi_1"] \arrow[rr, "\Phi=\Phi_1\oplus\Phi_2"] \arrow[dr, "\Phi_2"'] & & \mathcal{P}(\mathbb{N})/\operatorname{Fin} \\
& \mathcal{P}(A_2)/\operatorname{Fin} \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
$$

to assure that $\ker(\Phi) = \mathrm{Fin}$. Such Φ can never have a Baire-measurable lifting (because by the 0-1 law a maximal nonprincipal ideal is never Baire-measurable), but a (Baire-measurable) lifting of Φ_2 is a lifting of Φ on a "large" ideal $\mathcal{I}$.

To explain our reduction results for arbitrary homomorphisms we need some terminology. An ideal $\mathcal{I}$ is *ccc over* Fin if there is no uncountable almost disjoint (modulo Fin) family of $\mathcal{I}$-positive sets. This property is strictly between $\mathcal{I}$ being nonmeager and $\mathcal{P}(\mathbb{N})/\mathcal{I}$ being ccc. To $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ associate the ideal

$$\mathcal{J}_{\mathrm{cont}} = \{ A \,:\, \text{the restriction of } \Phi \text{ to } \mathcal{P}(A)/\mathrm{Fin} \text{ has a continuous lifting} \}.$$

The following was proved by Just ([17]) in the case of F_σ ideals (see also [16]).

Proposition 5.1. OCA *and* MA *imply that for every* $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ *the ideal* $\mathcal{J}_{\mathrm{cont}}$ *is ccc over* Fin *whenever* $\mathcal{I}$ *is an analytic P-ideal.*

Let us give a rough sketch of proof in the case of F_σ P-ideals. Fix an uncountable family $\mathcal{A}$ of almost disjoint subsets of integers. The first step consists in proving that Φ has a lifting whose graph is covered by countably many Borel functions on $\mathcal{P}(A)$ for all but countably many $A \in \mathcal{A}$. This is achieved by defining an open partition of $\mathcal{X} = \bigcup_{A\in\mathcal{A}} \mathcal{P}(A)$ such that (i) Φ has a Borel-measurable lifting on a K_1-homogeneous set (modulo going to a co-countable subset of $\mathcal{A}$) and (ii) an uncountable K_0-homogeneous set gives rise to a set whose image under Φ would have to fill a gap in the range, and then applying OCA to this partition. A diagonalization argument gives a continuous lifting on most members of $\mathcal{A}$, implying $\mathcal{J}_{\mathrm{cont}}$ is ccc over Fin.

To make the passage from F_σ to arbitrary analytic P-ideals, we need an additional definition. If φ induces $\mathcal{I}$ define a distance on the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ by

$$d_\varphi(A, B) = \lim_n \varphi((A \,\Delta\, B) \setminus [1, n]).$$

It turns out that $\langle \mathcal{P}(\mathbb{N})/\mathcal{I}, d_\varphi \rangle$ is a complete metric space. A construction analogous to the F_σ case now gives a continuous lifting of a 2^{-n}-approximation to Φ with respect to d_φ, and a d_φ-limit of these approximations is the desired Baire-measurable lifting. In this situation when we are working with approximations to a homomorphism instead of liftings, methods of Descriptive Set theory are used more substantially than in the F_σ case.

Let us note that if we replace "ccc over Fin" by "nonmeager" in the conclusion of Proposition 5.1, the obtained statement follows already from OCA. However, in the proof of the main result of this section, Theorem 5.2 below (in particular, in the proof that $\mathcal{J}_2$ is a P-ideal), it is crucial that the ideal $\mathcal{J}_{\mathrm{cont}}$ is ccc over Fin. A mapping F is an *almost lifting* of $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$

if it serves as a lifting of Φ on a ccc over Fin ideal. Most efforts of Chapter III of [8] are put in the proof of the following global version of Proposition 5.1.

THEOREM 5.2. OCA *and* MA *imply that every homomorphism between quotients over analytic P-ideals has a continuous almost lifting F.*

Let us start an outline of the proof of Theorem 5.2 by quoting two lemmas. The first one shows what is needed to make use of stabilizers (see the paragraph after Lemma 1.1) in a situation when we do not necessarily have a Baire-measurable lifting (it appears in [12] and [37] with an additional assumption that $\mathcal{H}$ is an ideal).

LEMMA 5.3. *If $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ is hereditary (i.e., closed under taking subsets of its elements), the following are equivalent:*

1. *$\mathcal{H}$ is a nonmeager subset of $\mathcal{P}(\mathbb{N})$,*
2. *every sequence $\{s_i\}$ of pairwise disjoint finite subsets of $\mathbb{N}$ has an infinite subsequence whose union is in $\mathcal{H}$.*

The following lemma connects the conclusion of Theorem 5.2 with the notion of an amalgamation of homomorphisms introduced earlier.

LEMMA 5.4. *If $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has a Baire-measurable almost lifting, then $\Phi = \Phi_1 \oplus \Phi_2$ so that Φ_2 has a continuous lifting and* $\ker(\Phi_1)$ *is ccc over* Fin.

Theorem 5.2 is proved by considering an extension of $\mathcal{J}_{\mathrm{cont}}$, the ideal

$$\mathcal{J}_2 = \{\, A : \Phi \restriction \mathcal{P}(A)/\operatorname{Fin} \text{ has a Baire-measurable almost lifting} \,\}.$$

The theorem clearly reduces to proving that $\mathcal{J}_2$ is an improper ideal, which requires uniformization of almost liftings Ψ_A ($A \in \mathcal{J}_2$), a task in which OCA has already proved to be very useful (see [40, Theorem 8.7], [48, p. 9]). By Lemma 5.4, we can assume that every Ψ_A is of the form Ψ_H (see Definition 2.1). These liftings are sufficiently additive to be coherent, namely ($\varphi_{\mathcal{I}}$ is a submeasure which induces $\mathcal{I}$)

$$(*) \qquad \lim_{n\to\infty} \sup_{\min s \geq n} \varphi_{\mathcal{I}}(\Psi_A(s) \,\Delta\, \Psi_B(s)) = 0.$$

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$ define $[\mathcal{J}_2]^2 = L_0^n(D) \cup L_1^n(D)$ by letting $\{A, B\} \in L_0^n$ if

$$\varphi_{\mathcal{I}}(\Psi_A(s) \,\Delta\, \Psi_B(s)) > 2^{-n}$$

for some $s \subseteq D$. Lower semicontinuity of $\varphi_{\mathcal{I}}$ implies this is an open partition.

LEMMA 5.5. *A set $D \subseteq \mathbb{N}$ is in $\mathcal{J}_2$ if and only if $\mathcal{J}_2$ is σ-$L_1^n(D)$-homogeneous for all n.*

PROOF. Assume $D \in \mathcal{J}_2$, and for $A \in \mathcal{J}_2$ let m_A be such that

$$\varphi_{\mathcal{I}}(\Psi_A(s) \,\Delta\, \Psi_D(s)) < 2^{-n-1}, \quad \text{for all } s \subseteq [m_A, \infty) \cap A \cap D$$

(such m_A exists by $(*)$). Then $\varphi_{\mathcal{I}}(\Psi_A(s) \,\Delta\, \Psi_B(s)) \leq 2^{-m}$ whenever $m_A = m_B$ and $s \subseteq [m_A, \infty) \cap D \cap A \cap B$, therefore each of the sets $\{\, A \in \mathcal{J}_2 : m_A =$

$\bar{m}\,\}$ ($\bar{m} \in \mathbb{N}$) can be partitioned into countably many $L_1^n(D)$-homogeneous sets, depending on the behavior of Ψ_A on the finite interval $[1, \bar{m})$.

Now assume there is a partition $\mathcal{J}_2 = \bigcup_i \mathcal{H}_i$ into $L_1^n(D)$-homogeneous sets. On each $\hat{\mathcal{H}}_i = \bigcup_{A \in \mathcal{H}_i} \mathcal{P}(A)$ we can define a Borel 2^{-n+2}-approximation Θ_i to Φ_*, a fixed lifting of Φ. Let s_n be a sequence which stabilizes Θ_i for each $i \in \mathbb{N}$. Applying Lemma 5.3, it is possible to find a continuous 2^{-n+30}-approximation to Φ_* on $\mathcal{J}_2$. A d_φ-limit of these approximations is the desired continuous almost lifting of Φ.

Therefore, if $\mathcal{J}_2$ is a proper ideal, then by OCA it has an uncountable $L_0^{\bar{n}}(\mathbb{N})$-homogeneous subset for some positive integer $\bar{n}$. Using this set and Lemma 5.5, one can define a partial ordering $\mathcal{P}_{\omega_1}$ which forces an uncountable almost disjoint family of $\mathcal{J}_2$-positive sets. This partial ordering is a descendant of Kunen's partial ordering for making gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ ccc-indestructible ([23]; see also [40, p. 74]), and its ccc-ness is assured by proving that $\mathcal{J}_2$ is a P-ideal. The fact that $\mathcal{J}_{\text{cont}}$ need not be a P-ideal is the main reason for introducing the more complex $\mathcal{J}_2$. Applying MA to $\mathcal{P}_{\omega_1}$ we obtain a witness that $\mathcal{J}_2$ is not ccc over Fin. Since $\mathcal{J}_{\text{cont}} \subseteq \mathcal{J}_2$, this contradicts Proposition 5.1 and ends the proof. $\dashv$

Theorem 5.2 readily implies yet another reduction of complexity: OCA and MA imply that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ embeds into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ if and only if $\mathcal{I} \leq_{RB} (\mathcal{J} \restriction A)$ (where $\mathcal{J} \restriction A = \{\, B \subseteq A \,:\, B \in \mathcal{J} \,\}$) for some $\mathcal{J}$-positive set A (take $A = F(\mathbb{N})$). This preordering does differ from $\leq_{RB}$. For example, for any pair $\mathcal{I}_f, \mathcal{I}_g \not\approx \operatorname{Fin}$ of summable ideals there is an $\mathcal{I}_g$-positive set A such that $\mathcal{I}_F \leq_{RB} \mathcal{I}_g \restriction A$. However, we have $\mathcal{I}_{1/\sqrt{n}} \not\leq_{RB} \mathcal{I}_{1/n}$. Theorem 5.2 also implies the following metatheorem.

Proposition 5.6. *For analytic P-ideals $\mathcal{I}$ and $\mathcal{J}$ the following are equivalent*:
1. *It is not provable in* ZFC *that $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$.*
2. OCA *and* MA *imply that $\mathcal{P}(\mathbb{N})/\mathcal{I} \not\approx \mathcal{P}(\mathbb{N})/\mathcal{J}$.*
3. *The quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are not Baire-isomorphic.*

Proof. Since the ideals $\mathcal{I}$ and $\mathcal{J}$ are $F_{\sigma\delta}$ and every Baire-measurable lifting can be turned into a continuous one, statement 3 is $\mathbf{\Sigma}_2^1$ and therefore absolute by Shoenfield's theorem. Let us prove only the nontrivial direction, 3 implies 2. So assume OCA and MA and let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be an isomorphism with lifting Φ_*. A continuous almost lifting F of Φ guaranteed by Theorem 5.2, is a lifting of a homomorphism $\Phi_1\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(F(\mathbb{N}))/\mathcal{J}$. Let B be such that $\Phi_*(B) = \mathbb{N} \setminus F(\mathbb{N})$. Since $B \in \ker(\Phi_1)$, the analytic ideal $\ker(\Phi) \restriction B$ is nonmeager, and therefore improper, so F is a continuous lifting of Φ. $\dashv$

Corollary 5.7. OCA *and* MA *imply that quotients over the ideals* Fin, $\emptyset \times \operatorname{Fin}$, $\mathcal{I}_{1/n}$, $\mathcal{I}_{1/\sqrt{n}}$, $\mathcal{Z}_0$, $\mathcal{Z}_{\log}$ *are pairwise nonisomorphic.*

We should remark that the "local" conclusion of Proposition 5.1 is sufficient to prove that quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are nonisomorphic, but via a substantially more involved argument (see [15]).

A straightforward consequence of Theorem 5.2 and the Radon-Nikodym property is that under OCA and MA the automorphism group $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I})$ of a quotient over a nonpathological ideal coincides with its subgroup of trivial automorphisms. To formulate another consequence of Theorem 5.2, we need to introduce some terminology. Two families $\mathcal{A}$ and $\mathcal{B}$ of sets of integers form a *gap* if they are *orthogonal* ($A \cap B$ is finite for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$) and they can not be *separated* (no $C \subseteq \mathbb{N}$ almost includes all members of $\mathcal{A}$ and is almost disjoint from all members of $\mathcal{B}$). The following result was suggested by Todorcevic, and it is related to his results about analytic gaps ([41], [42]).

COROLLARY 5.8. *Under* OCA *and* MA, *if $\mathcal{I}$ is an arbitrary ideal and there is a monomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$, then $\mathcal{I}, \mathcal{B}$ is not a gap for any $\mathcal{B}$.*

PROOF. If Φ_* is a c. a. lifting of Φ then set $\{\, n : \Phi_*(\{n\}) \neq \emptyset \,\}$ separates $\mathcal{I}$ from $\mathcal{B}$. ⊣

§6. Gaps in analytic quotients. One of the most beautiful early discoveries in the study of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ was Hausdorff's construction of a *gap* inside this quotient, namely a pair of orthogonal, σ-directed under the inclusion mod finite, families which can not be separated (see [32]). We shall call any such pair a *Hausdorff gap*. A remarkable feature of Hausdorff's gap is the fact that its construction did not require any additional set-theoretic axioms, a phenomenon which is rarely encountered at this level of complexity in investigating $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Both sides of this gap have length ω_1, and after the invention of Forcing, Kunen ([23]) proved another surprising fact about gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$: The ordinary axioms of Set Theory are not sufficient to give us any other types of Hausdorff gaps. (Later Todorcevic ([40]) deduced the same conclusion from OCA, and the partition he used in that deduction is an ancestor of Velickovic's partition used in proof of Proposition 5.1, tailored so that an uncountable K_0-homogeneous set gives rise to a gap in the range and an element in the domain whose image is supposed to fill this gap.)

The question of the existence of gaps in arbitrary analytic quotients was addressed only relatively recently by Just ([32]). Mazur ([28]) constructed $\langle\omega_1, \omega_1\rangle$-gaps in every quotient over an F_σ ideal, and, using CH, in every analytic quotient. Then Todorcevic ([39], [42]) proved the following:

THEOREM 6.1. *Every monomorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over an analytic ideal with a Baire-measurable lifting preserves all Hausdorff gaps. If $\mathcal{I}$ is moreover a P-ideal, then Φ preserves all gaps.*

(Theorem 6.1 was an initial motivation for stating SLC, for its proof is substantially simpler when there is a c. a. lifting.) By a theorem of Mathias ([27]), the quotient $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is Baire-embeddable into every analytic quotient, so the gap-spectrum of every analytic ideal includes that of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. It is natural to ask whether analogues of Kunen's or Todorcevic's theorem are true for other analytic quotients (see [39, Problem 2], [42, Problems 3 and 4]).

PROPOSITION 6.2. *The gap-spectra of* $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\emptyset\times\operatorname{Fin}$ *and* $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *coincide.*

PROOF. Let $\mathcal{A},\mathcal{B}$ be a gap in $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\emptyset\times\operatorname{Fin}$. If for every n there is $C_n\subseteq\{n\}\times\mathbb{N}$ which separates restrictions of $\mathcal{A}$ and $\mathcal{B}$ to $\{n\}\times\mathbb{N}$, then $C=\bigcup_n C_n$ separates $\mathcal{A}$ and $\mathcal{B}$. Therefore, the restrictions of $\mathcal{A}$ and $\mathcal{B}$ to some $\{n\}\times\mathbb{N}$ already form a gap, which can be assured to be of the same type as the original one. ⊣

PROPOSITION 6.3. *Every monomorphism of* $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\emptyset\times\operatorname{Fin}$ *into a quotient* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *over an analytic ideal with a Baire-measurable lifting preserves all Hausdorff gaps. If* $\mathcal{I}$ *is moreover a P-ideal, then it preserves all gaps.*

The quotient $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\operatorname{Fin}\times\emptyset$ can be considered as an inverse limit

$$\mathcal{P}(\mathbb{N}\times\mathbb{N})/\operatorname{Fin}\times\emptyset=\varprojlim\langle\mathcal{P}(\Gamma_f)/\operatorname{Fin},\pi^f_g,\mathbb{N}^{\mathbb{N}}\rangle,$$

where $\Gamma_f=\{\langle m,n\rangle : n\leq f(m)\}$ and $\pi^f_g:\Gamma_f/\operatorname{Fin}\to\Gamma_g/\operatorname{Fin}$ (for $f\geq g$) is the restriction mapping. The proof of Proposition 6.2 can be applied to $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\operatorname{Fin}\times\emptyset$ using the following consequence of OCA:

(A) For every Hausdorff gap $\mathcal{A},\mathcal{B}$ in $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\operatorname{Fin}\times\emptyset$ there is $f\in\mathbb{N}^{\mathbb{N}}$ such that the restriction of $\mathcal{A}$, $\mathcal{B}$ to Γ_f is a gap.

Here is a more familiar consequence of (A):

(A′) If functions $g_f:\Gamma_f\to\mathbb{N}$ $(f\in\mathbb{N}^{\mathbb{N}})$ are pairwise equal modulo Fin, then there is a single $h\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that each g_f is equal to $h\restriction\Gamma_f$ modulo Fin.

This statement is well-studied, in particular in connection with Milnor's additivity axiom for strong homology (see [26], [5], [40, 8.7], [38]).

PROPOSITION 6.4. *Assume* OCA *or* (A). *Then*:

a. *The gap-spectra of* $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\operatorname{Fin}\times\emptyset$ *and* $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *coincide.*

b. *Every monomorphism of* $\mathcal{P}(\mathbb{N}\times\mathbb{N})/\operatorname{Fin}\times\emptyset$ *into a quotient* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *over an analytic ideal with a Baire-measurable lifting preserves all Hausdorff gaps.*

(By Theorem 8 of [39], $\mathcal{I}$ as in (b) can not be a P-ideal.) Perhaps the additional assumptions in Proposition 6.4 are unnecessary. However, the gap-spectrum of a quotient over an analytic ideal generated by $\emptyset\times\operatorname{Fin}$ and $\operatorname{Fin}\times\emptyset$ can be different from that of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ in some forcing extension (see [8, §26]). It is also interesting that a version of Theorem 6.1 fails for this ideal.[1]

§7. Weak extension principle. The study of homomorphisms of analytic quotients and their liftings was initiated by a question in Topology: Is it consistent that all automorphisms of the Čech-Stone remainder $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ (i.e., the Stone space of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$) are induced by mappings of $\mathbb{N}$ (or *trivial*)? The motivation for asking this question came from a result of W. Rudin ([31]) to the effect that under CH every two P-points in $\mathbb{N}^*$ can be mapped one in another by an automorphism of $\mathbb{N}^*$. Since in this situation the number of P-points is $2^{2^{\aleph_0}}$ while the number of c. a. liftings is only $2^{\aleph_0}$, there are nontrivial automorphisms of $\mathbb{N}^*$. The above question is equivalent to the question whether all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ can be trivial, which was answered in positive by Shelah ([33]).

By another classical result of Parovičenko ([30]), CH implies that $\mathbb{N}^*$ maps onto any of its finite powers and onto α^* for every countable limit ordinal α. It was shown by Just ([14], [17]) and Dow-Hart ([4]) that both these statements are false under OCA and MA. We shall systematize and extend these results, by using a *weak Extension Principle*, wEP. A word about the notation—we shall write $X^{(d)}$ instead of X^d for the dth power of space X, in order to distinguish the ordinal ω^2 from the square $\omega^{(2)}$. A map $F\colon (\alpha^*)^{(d)} \to (\gamma^*)^{(l)}$ is *trivial* if there is is $h\colon \alpha^{(d)} \to \gamma^{(l)}$ such that F is the restriction to $(\alpha^*)^{(d)}$ of the unique extension of h to $(\alpha^*)^{(d)}$.

(wEP) Assume α and γ are countable ordinals. Then for a continuous $F\colon (\alpha^*)^{(d)} \to (\gamma^*)^{(l)}$ there is a clopen partition $(\alpha^*)^{(d)} = U_0 \cup U_1$ such that $F''U_0$ is nowhere dense and $F \restriction U_1$ is trivial.

An extension of our results about liftings implies the following.

Theorem 7.1. OCA *and* MA *imply* wEP.

Since a map $F\colon (\beta\alpha)^{(d)} \to (\beta\gamma)^{(l)}$ is uniquely determined by its restriction to $\alpha^{(d)}$, wEP reduces a question of whether $(\alpha^*)^{(d)}$ maps onto $(\gamma^*)^{(l)}$ to an (absolute) statement about the existence of a map from $\alpha^{(d)}$ into $\gamma^{(l)}$ with some simple properties. Since every ordinal α can be written in a unique way as $\alpha_0 + \alpha_1$ so that α_1 is *indecomposable* (i.e. $\alpha_1 = \alpha_2 + \alpha_3$ implies $\alpha_1 = \alpha_3$), and since this implies $\alpha^* = \alpha_1^*$, we can restrict our attention to indecomposable ordinals. Recall that a continuous map is *perfect* if and only if preimages of compact sets are compact (see [6]).

Theorem 7.2. *Under* wEP *the following are equivalent for indecomposable countable ordinals* α *and* γ:

1. $\gamma \leq \alpha$.
2. *There is a continuous perfect surjection* $h\colon \alpha \to \gamma$.

[1] Added in proof. Theorem 6.1 even fails for some F_σ P-ideals.

3. α^* *maps onto* γ^*, *or in symbols*, $\alpha^* \twoheadrightarrow \gamma^*$.

A special case $\alpha = \omega$ and $\gamma = \omega^2$ of Theorem 7.2 was proved by A. Dow and K. P. Hart ([4]), who used it to prove that $\mathbb{N}$ is essentially the only locally compact, σ-compact space whose Čech-Stone remainder is a surjective image of $\mathbb{N}^*$.

Clearly, wEP implies that every automorphism of $(\mathbb{N}^*)^{(d)}$ ($d \in \mathbb{N}$) is induced by some $h\colon \mathbb{N}^{(d)} \to \mathbb{N}^{(d)}$. This leaves open the question which of these maps h induce automorphisms. The simplest way to induce an automorphism is to take permutations $\varphi_1, \dots, \varphi_d$ of $\mathbb{N}$ and π of $\{1, \dots, d\}$, and define $h\colon \mathbb{N}^{(d)} \to \mathbb{N}^{(d)}$ by

$$h(n_1, \dots, n_d) = (\varphi_1(n_{\pi(1)}), \dots, \varphi_d(n_{\pi(d)})).$$

Not all automorphisms of $(\beta\mathbb{N})^{(d)}$ are induced in this way—for any positive integer m we can split $\mathbb{N}^{(d)}$ into m many disjoint rectangles in two ways, say $\mathbb{N}^{(d)} = R_1 \cup \dots \cup R_m$ and $\mathbb{N}^{(d)} = P_1 \cup \dots \cup P_m$, and define h so that it maps R_i into P_i and that on each R_i it is of the above form. Let us say that an automorphism of $(\beta\mathbb{N})^{(d)}$ is *trivial* if it is induced in this way.

THEOREM 7.3. *Under* wEP *all automorphisms of* $(\mathbb{N}^*)^{(d)}$ *are trivial.*

§8. Remarks and questions.

Let us point out to, in our opinion, the most important open problems in this area.

QUESTION 8.1. Is it true that every *isomorphism* of quotients over analytic P-ideals which has a Baire-measurable lifting has a c.a. lifting of the form Φ_h?

A positive answer would give a satisfactory answer to our basic question in the realm of analytic P-ideals: Two quotients over analytic P-ideals would be isomorphic if and only if the corresponding ideals are isomorphic. This question has a finitary version, similar to statement of Theorem 3.1 (see [10, Question 10]).

In §5 we have seen that there are homomorphisms $\Phi\colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ without a c. a. lifting. However, to our best knowledge every homomorphism that can be constructed without the trivializing effect of CH has an additive lifting.

QUESTION 8.2. Do OCA and MA imply that every homomorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ has an additive lifting?

A positive answer is equivalent to a *strong Extension Principle*: every continuous map from $\mathbb{N}^*$ into $\mathbb{N}^*$ extends to $\beta\mathbb{N}$, and it would greatly improve our understanding of closed images of $\mathbb{N}^*$.

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