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FROM STABILITY TO SIMPLICITY

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§1. Introduction. In this report we wish to describe recent work on a class of first order theories first introduced by Shelah in [32], the *simple* theories. Major progress was made in the first author's doctoral thesis [17]. We will give a survey of this, as well as further works by the authors and others.

The class of simple theories includes stable theories, but also many more, such as the theory of the random graph. Moreover, many of the theories of particular algebraic structures which have been studied recently (pseudofinite fields, algebraically closed fields with a generic automorphism, smoothly approximable structures) turn out to be simple. The interest is basically that a large amount of the machinery of stability theory, invented by Shelah, is valid in the broader class of simple theories. Stable theories will be defined formally in the next section. An exhaustive study of them is carried out in [33]. Without trying to read Shelah's mind, we feel comfortable in saying that the importance of stability for Shelah lay partly in the fact that an unstable theory T has 2^λ many models in any cardinal $\lambda \geq \omega_1 + |T|$ (proved by Shelah). (Note that for $\lambda \geq |T|$, 2^λ is the maximum possible number of models of cardinality λ.) Of course the fact that nice theorems could be proved under the assumption of stability was also something important. (The instability/stability dichotomy is one of those "useful dichotomies" which often arise in Shelah's work.) Our point is that historically it was the "counting models" problematic which gave rise to the notion of stability. Morley proved that if T is a countable theory, and T has a unique model in some uncountable cardinal then T has a unique model in every uncountable cardinal. Part of his proof was to show that if T satisfies the hypothesis, then T is ω-stable, that is over any countable model M of T there are only countably many complete types. ω-stability of a theory has consequences such as the existence of prime models over arbitrary sets and the existence of a dimension theory (Morley rank), all of which played important roles in Morley's proof. For T a complete theory and λ a cardinal, let $I(\lambda, T)$ be the number of models of T of cardinality λ, up to isomorphism. Shelah's

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program was to determine, at least for countable theories T, the possible functions $I(-,T)$ restricted to uncountable arguments. So from the point of view of this program, one could assume T to be stable. Under the assumption of stability, a complex and highly nontrivial machinery was developed in order to analyze models, build models, distinguish or identify isomorphism types, and, as it turned out, to shed light on algebraic examples. Shelah's program was eventually highly successful, although there were a series of further dichotomies which had to be made: superstable/non-superstable, NDOP/DOP, NOTOP/OTOP.

There was a widespread belief that the machinery developed by Shelah only applied in the very restricted context of stable theories, and this was more or less considered as just the way the world was. As we will point out in this survey, this complex theory turns out to be relevant in a much larger context, one which, moreover, includes many "generic" structures (in the sense of Robinson), as well as interesting algebraic situations. Of course there have been other reasonably successful attempts to develop a general model theory of certain unstable structures, one such being the theory of O-minimality. It is however clear, that whatever the power of O-minimality, it is a highly specific theory without the generality of stability theory. (So we are in a sense waiting for a theory which bears the same relation to O-minimality that stability bears to strong minimality.)

The key notion behind simplicity (and even stability) is Shelah's notion of forking, which should be considered as a combinatorial (rather than topological) account of dimension theory, and as such is a profound insight into mathematics. Roughly speaking, a complete type $p(x)$ is said to fork over a set A of parameters, if some infinite A-indiscernible sequence $(p_i(x))_i$ of A-conjugates of $p(x)$ is inconsistent (where by an A-conjugate of p we mean an image of p under an automorphism which fixes A pointwise). For sets $A \subset B$ of parameters, a is said to be independent from B over A if $\mathrm{tp}(a/B)$, the type of a over the parameter set B, does not fork over A. The intuition is that there are no more first order connections between a and B than those coming from the first order connections between a and A. The key property of simplicity (of a complete theory T) is that (working in a big saturated model of T), for any complete type $p(x)$ over a set B there is a subset A of B of cardinality at most that of T such that p does not fork over A. It turns out that T is stable just if T is simple, and for any $A \subset B$, the set of complete types $p(x)$ over B which do not fork over A and have a fixed restriction to A has cardinality at most $2^{|T|}$. At the technical level, one immediately has the impression of a distinction between the theory of independence (forking) and the theory of multiplicity. In many of the accounts of stability theory (including the second author's first book [27], which expounded the insights of the French school of Lascar and Poizat [26]), there was a certain confusion between these two aspects of the theory.

One of the advances coming out of our study of simple theories is the proper separation of these aspects, in such a way that even the theory of forking in stable theories becomes substantially clearer. In section 2 we will outline the theory of forking in simple theories, as well as analogues of multiplicity theory. In section 3 we give some examples of simple theories. In section 4 we introduce the important idea of a Lascar strong type. In section 5 we mention definable groups in simple theories and how stable group theory generalizes. In section 6 the existence of "canonical bases" for types in stable theories is generalized to simple theories. In section 7 we mention some problems.

The question arises, in this post-classification theory period, of what, if anything, is the aim or final purpose of "pure" model theory. Of course there remain difficult problems within the classification theory problematic, such as Vaught's conjecture (that a countable complete theory has either countably many or continuum many countable models), the classification of ω_1-saturated models of stable theories, and classification theory for uncountable theories. On the other hand we are seeing at the present time unprecendented levels of interaction between model theory and what some may call "core" mathematics (an example being Hrushovski's work in diophantine geometry). We find that many of the notions thrown up by stability theory (such as orthogonality, coordinatization, regular types, generic types of definable groups) have meaning and implications in various mathematical contexts, such as number theory and differential equations. The validity of many of these notions outside the stable theories context (such as the work on simple theories being discussed in this article) is of great importance. So let us mention some general themes which could inform future work. One is dimension theory, the search for general notions of independence. Another theme is the manner in which a structure can be built up from or "coordinatized" by "simpler" structures. The classical stability-theoretic context for this is Shelah's theory of semi-regular types. But note that the work of Ax-Kochen, whereby certain valued fields are determined in various ways by their value groups and residue class fields, also fits into this theme. One more theme is the classification of first order theories. Note that arithmetic (the theory of the ring of integers, or the field of rationals) is in all possible senses "wild", although the search for an understanding of it is behind much mathematics. It is rather interesting that the auxiliary structures which mathematicians have used or sometimes even invented to help understand arithmetic, such as the local fields $\mathbf{R}$ and $\mathbf{Q}_p$, and the finite fields $\mathbf{F}_q$, are key unstable structures supporting dimension theories which model theorists are currently generalizing.

§2. Forking. As Shelah's notion of forking is central to the subject of this survey, we will give the formal definition, which is actually not so obscure.

Let us first fix (standard) model theoretic notation. T will denote a complete theory with no finite models, in a language L, and we work in a saturated model $\mathbf{C}$ of T of cardinality κ for some large κ. It is sometimes convenient to assume that κ is strongly inaccessible. For a first order sentence σ with parameters from $\mathbf{C}$, the expression $\models \sigma$ means that σ is true in $\mathbf{C}$. A, B, etc., will denote subsets of $\mathbf{C}$ of cardinality strictly less than κ. Similarly M, N, etc., will denote elementary substructures of $\mathbf{C}$ of cardinality strictly less than κ. A partial type over A is a set of formulas with parameters from A which is realized in $\mathbf{C}$. If we write such a partial type as $\Phi(\bar{x})$ we mean that the free variables in the formulas are among the tuple $\bar{x}$. Generally $\bar{x}$, $\bar{y}$, etc., denote possibly infinite tuples of variables (but of length strictly less than κ). Similarly for tuples $\bar{a}$, $\bar{b}$ from $\mathbf{C}$. However in most cases the reader can take these to be finite tuples. The notions algebraic closure and definable closure are rather important. b is said to be in the algebraic closure of A, $b \in \mathrm{acl}(A)$, if there is some formula $\phi(x)$ with parameters from A such that $\models \phi(b)$, and $\phi(x)$ has only finitely many other solutions in $\mathbf{C}$. If, moreover, b is the unique solution of $\phi(x)$, we say that b is in the definable closure of A, $b \in \mathrm{dcl}(A)$. Similarly for finite tuples $\bar{b}$ in place of elements. acl and dcl have an automorphism-theoretic interpretation: $b \in \mathrm{acl}(A)$ ($b \in \mathrm{dcl}(A)$) if b has only finitely many images (exactly one image) under automorphisms of $\mathbf{C}$ which fix A pointwise. The notion of an indiscernible sequence is pervasive in the subject, so we recall the definition. Let $A \subset \mathbf{C}$, let I be a totally ordered set, and let $(\,\bar{b}_i : i \in I\,)$ be a sequence of tuples from $\mathbf{C}$ of some fixed length. We will say that $(\,\bar{b}_i : i \in I\,)$ is indiscernible over A, if for any $n < \omega$ and $i_1, \dots, i_n \in I$, $\mathrm{tp}(\bar{b}_{i_1}, \dots, \bar{b}_{i_n}/A)$ depends only on the order type of $(i_1, \dots, i_n)$. Usually I will be an infinite ordinal (typically ω). If $\bar{a}$ is a tuple from $\mathbf{C}$ then $\mathrm{tp}(\bar{a}/A)$ denotes the set of all formulas $\phi(\bar{x})$ with parameters from A which are true of $\bar{a}$. This is a complete type over A and all complete types arise in this way. $S(A)$ denotes the set of all complete types over A. Let $\Phi(\bar{x}, \bar{b})$ be a partial type over the (possibly infinite) tuple $\bar{b}$. We will say that $\Phi(\bar{x}, \bar{b})$ divides over a set A if there is some infinite A-indiscernible sequence $(\,\bar{b}_i : i < \omega\,)$ of realizations of $\mathrm{tp}(\bar{b}/A)$ such that $\bigcup\{\, \Phi(\bar{x}, \bar{b}_i) : i < \omega \,\}$ is inconsistent. We say that $\Phi(\bar{x}, \bar{b})$ forks over A if Φ implies some finite disjunction of formulas, each of which divides over A. In the relevant situations, forking will be equivalent to dividing. The notion of forking (as opposed to just dividing) is introduced for certain technical reasons. We say that $\bar{a}$ is independent from B over A if $\mathrm{tp}(\bar{a}/A \cup B)$ does not fork over A. In certain situations, this notion reduces to familiar ones. For example, let T be the theory of algebraically closed fields of some fixed characteristic (in the language of rings). Let $F < K$ be subfields of $\mathbf{C}$. Then $\bar{a}$ is independent from K over F if the transcendence degree of $K(\bar{a})$ over K equals that of $F(\bar{a})$ over F.

One might wonder in which contexts this notion of independence is "well-behaved". Motivated by the problem of how many models T has in various cardinalities, Shelah studied stable theories. T is said to be stable if for any model M of T the number of complete types (in finitely many variables) over M has cardinality at most $|M|^{|T|}$. Equivalently there are no formula $\phi(\bar{x}, \bar{y})$ and tuples $\bar{a}_i, \bar{b}_i$ for $i < \omega$ such that $\models \phi(\bar{a}_i, \bar{b}_j)$ if and only if $i < j$. Shelah [33] proves:

PROPOSITION 2.1. *Suppose T is stable. Then the following hold*:

(i) *For any $p(\bar{x}) \in S(A)$ (where $\bar{x}$ is a finite tuple) there is a subset A_0 of A of cardinality at most $|T|$ such that p does not fork over A_0,*
(ii) *For any $p(\bar{x}) \in S(A)$ and $B \supset A$, there is $q(\bar{x}) \in S(B)$ such that $p \subset q$ and q does not fork over A. (We call q a nonforking extension of p over B.)*
(iii) *For any $A \subset B$ and tuple $\bar{a}$, $\mathrm{tp}(\bar{a}/B)$ does not fork over A if and only if for each tuple $\bar{b}$ from B, $\mathrm{tp}(\bar{b}/A \cup \{\bar{a}\})$ does not fork over A.*
(iv) *For any $A \subset B \subset C$ and $\bar{a}$, $\mathrm{tp}(\bar{a}/C)$ does not fork over A if and only if $\mathrm{tp}(\bar{a}/C)$ does not fork over B and $\mathrm{tp}(\bar{a}/B)$ does not fork over A.*
(v) *If M is a model and $p(\bar{x}) \in S(M)$ then for any $B \supset M$, p has a unique nonforking extension over B.*

In fact the reader should see that (i) to (v) above in turn imply stability of T. (This can be seen by counting types.)

With a little imagination, one can consider (i) to (iv) as being properties of a "universal-algebraic" nature, whereas (v) is something different. We call a type $p(\bar{x})$ *stationary* if p has a unique nonforking extension over any $B \supset A$. A very important result in basic stability theory, is that for any A and $p(\bar{x}) \in S(A)$, the additional data telling us for each A-definable equivalence relation E with finitely many equivalence classes, in which class $\bar{x}$ lies, uniquely determines any nonforking extension of $p(\bar{x})$ to any B. We call such equivalence relations, finite equivalence relations over A. Model theory has a convenient way of viewing this data as a complete type; we introduce new "sorts" for $\emptyset$-definable equivalence relations (the elements of the sort being the equivalence classes), and then for any set A, $\mathrm{acl}^{\mathrm{eq}}(A)$ denotes the set of elements from all these new sorts which are in the algebraic closure of A (namely have only finitely many images under automorphisms of $\mathbf{C}$ which fix A pointwise), and then define $\mathrm{stp}(\bar{a}/A)$ (the strong type of $\bar{a}$ over A), to be $\mathrm{tp}(\bar{a}/\mathrm{acl}^{\mathrm{eq}}(A))$. The finite equivalence relation theorem can then be restated as

(v)$'$ *Any strong type is stationary.*

Associated with (v) (and (v)$'$) is the theory of *multiplicity* and *definability* in stable theories, which we now see as something different from the theory of forking per se. Characteristic features of this kind are

(i) Definability of types: if $p(\bar{x})$ is a complete type over a model M, then for each formula $\phi(\bar{x}, \bar{y})$ of L, the set of $\bar{b}$ from M such that $\phi(\bar{x}, \bar{b}) \in p$ is defined by a formula $\psi_\phi(\bar{y})$ with parameters from M.

(ii) Open Mapping Theorem: for any model M and set B containing M, the mapping from $S(M)$ to $S(B)$ taking p to its unique nonforking extension, is a continuous map in the Stone space topology.

In [32], Shelah attempted a generalization of the above theory.

DEFINITION 2.2. The theory T is said to be simple if for any complete type $p(\bar{x}) \in S(A)$ there is a subset A_0 of A of cardinality at most $|T|$ such that p does not fork over A_0.

So one takes as the definition of simplicity, property (i) in Proposition 2.1 above, which we know to hold for stable theories. Shelah in [32] pointed out several equivalences of the above definition. These are not altogether trivial, and the ranks $D(-, \Delta, k)$ which we will describe later, intervene in the proofs.

FACT 2.3. *T is simple if and only if there do not exist an L-formula $\phi(x, \bar{y})$ (so x is a single variable), and tuples $\bar{b}_\tau$ for $\tau \in \omega^{<\omega}$ such that,*

(i) *for each τ, $\{\phi(x, \bar{b}_{\tau\frown i}) : i < \omega\}$ is pairwise inconsistent, and*
(ii) *for each $\eta \in \omega^\omega$, $\{\phi(x, \bar{b}_{\eta\lceil i}) : i < \omega\}$ is consistent.*

FACT 2.4. *T is simple if and only if for any complete 1-type $p(x) \in S(A)$, there is a subset A_0 of A of cardinality at most $|T|$ such that $p(x)$ does not divide over A_0.*

Both these above facts are useful in verifying that particular theories are simple, or not simple, as the case may be. For example the simplicity of the theory of the random graph (see Example 6.2) can be deduced quite directly from Fact 2.4. The nonsimplicity of any theory with a definable total ordering can also be deduced from Fact 2.3 (as also any theory with the *strict order property*). More details will be given in Section 6.

Shelah in [32] did some fundamental work on trying to generalize stability-theory to the simple theories context, with many new ideas and techniques. He pointed out to start with:

LEMMA 2.5. *Let T be simple. Then* (ii) *of Proposition* 2.1 *holds.*

In his paper there were some approximations to (iii) and (iv) of Proposition 2.1 for simple theories, but not the full statements. The first author, in his doctoral thesis [17], managed to obtain (iii) and (iv) by direct arguments. An important notion borrowed from stability theory is that of a *Morley sequence*. We will use henceforth a, b, etc., and x, y, etc., to denote arbitrary finite tuples of elements or variables. Let $p(x)$ be a complete type over A. By a Morley sequence in p of length α we mean an A-indiscernible sequence $(b_i : i < \alpha)$ of realizations of p such that $\mathrm{tp}(b_i/A \cup \{b_j : j < i\})$ does not fork over A for each $i < \alpha$. For simple T, infinite Morley sequences in any complete type exist. The crucial breakthrough of the first author was the following ([17] and [18]):

LEMMA 2.6. *Let T be simple, let $\phi(x,b)$ be a formula, and A a set of parameters. Then the following are equivalent*:

(a) *$\phi(x,b)$ divides over A*,
(b) *for some Morley sequence $(b_i : i < \omega)$ in $\mathrm{tp}(b/A)$, $\{\phi(x,b_i) : i < \omega\}$ is inconsistent*,
(c) *for any Morley sequence $(b_i : i < \omega)$ in $\mathrm{tp}(b/A)$, $\{\phi(x,b_i) : i < \omega\}$ is inconsistent*,
(d) *$\phi(x,b)$ forks over A.*

COROLLARY 2.7. *Let T be simple. Then* (i), (ii), (iii) *and* (iv) *of Proposition* 2.1 *hold.*

(i) is the definition. (ii) is by Shelah. We briefly explain (iii). Suppose $\mathrm{tp}(b/A \cup \{a\})$ does not fork over A. By (i) and some standard methods for obtaining indiscernibles, we can find a sequence $(b_i : i < \omega)$ which is indiscernible over $A \cup \{a\}$ and such that $b_0 = b$ and for each i, $\mathrm{tp}(b_i/A \cup \{a\} \cup \{b_j : j < i\})$ does not fork over A. In particular $(b_i : i < \omega)$ is a Morley sequence in $\mathrm{tp}(b/A)$. On the other hand, for each formula $\phi(x,y)$ over A which holds of (a,b), we see that a realizes $\phi(x,b_i)$ for all i. By the (d) implies (c) part of the above lemma, $\mathrm{tp}(a/A \cup \{b\})$ does not fork over A.

With the above Corollary, the "universal-algebraic" part of forking theory for stable theories, is seen to generalize totally to the simple theory context. With only these tools, the first author was able to generalize Lachlan's theorem ([22]) on the number of countable models of a superstable theory to the case of supersimple theories. We will call T supersimple if every complete type does not fork over some *finite* subset of the domain of the type. T is superstable if it is stable and supersimple. The first author proved [20]:

THEOREM 2.8. *Let T be a countable supersimple theory which is not ω-categorical. Then T has an infinite number of countable models, up to isomorphism.*

The classical proof for superstable theories uses definability theory (specifically the open mapping theorem) in an essential way. The first author showed that the proof can be done using only properties (i) to (iv) (and the definition of forking). We give some details. $\mathrm{tp}(b/A)$ is said to be semi-isolated if there is some formula $\phi(x) \in \mathrm{tp}(b/A)$ which implies $\mathrm{tp}(b/\emptyset)$. The first author proves (for simple T) that if a, b realise the same complete type over $\emptyset$, and $\mathrm{tp}(b/a)$ is semi-isolated but $\mathrm{tp}(a/b)$ is not semi-isolated, then $\mathrm{tp}(b/a)$ forks over $\emptyset$. This replaces the use of the open mapping theorem, and allows the rest of Lachlan's proof to go through.

The question arises of what kind of analogue of property (v) could hold in simple theories. Hrushovski, in [12], [11] and [14], as part of his analyses of smoothly approximable structures and pseudofinite fields, isolated an important property of independence, which he called "The Independence Theorem". Cherlin has suggested "Amalgamation of types" as a better name. In any case, motivated partly by this, and partly by Shelah's work, the authors were able to show in [21]:

THEOREM 2.9. *Let T be simple. Let M be a model and $p(x)$ a complete type over M. Let $A \supset M$ and $B \supset M$ be such that A is independent from B over M (that is,* $\mathrm{tp}(A/B)$ *does not fork over M). Let $p_1(x) \in S(A)$ and $p_2(x) \in S(B)$ be nonforking extensions of $p(x)$. Then $p_1(x)$ and $p_2(x)$ have a common nonforking extension $q(x)$ over $A \cup B$.*

We will call this theorem the "Independence Theorem over a model", and we also number it as property (vi). We will not even describe the proof. But suffice it to say, that infinitary stable formulas intervene. One may ask what is so important about M being a model here. At the technical level, this is used to produce an infinite Morley sequence I of $p(x)$ with the property that whenever $\mathrm{tp}(J/M) = \mathrm{tp}(I/M)$ then there is an infinite sequence I' such that both $I^\frown I'$ and $J^\frown I'$ are Morley sequences in p. Note that for stable theories (vi) is an immediate consequence of (v) and the other properties. (The unique nonforking extension of p over $A \cup B$ must extend both p_1 and p_2.) In [21] we prove:

THEOREM 2.10. *Suppose that the T supports some notion of independence satisfying properties* (i), (ii), (iii), (iv) *and* (vi) *then T must be simple and this notion of independence must coincide with nonforking.*

So property (vi) is intimately connected with the notion of forking.

Let us now mention ranks. Ranks of one kind or another played a crucial role in Shelah's treatment of stability. We referred briefly above to a certain rank $D(\Sigma(x), \phi(x, y), k)$, introduced by Shelah. Here $\Sigma(x)$ is some partial type over some set, $\phi(x, y)$ is an L-formula, and k a natural number. The crucial inductive clause in the definition is: $D(\Sigma(x), \phi(x, y), k) \geq n + 1$ if there is some infinite set $\{ b_i : i < \omega \}$ such that $\{ \phi(x, b_i) : i < \omega \}$ is k-inconsistent (that is, every k-size subset is inconsistent), and $D(\Sigma(x) \cup \{\phi(x, b_i)\}, \phi(x, y), k) \geq n$ for all i. We then have the following, part (a) being due to Shelah, and part (b) to the authors.

PROPOSITION 2.11.

(a) *T is simple if and only if for all $\phi(x, y) \in L$ and $k < \omega$,*

$$D(x = x, \phi(x, y), k) < \omega.$$

(b) *Assume T to be simple. Let $A \subset B$, $p(x) \in S(A)$, and let $q(x) \in S(B)$ be an extension of p. Then q does not fork over A if and only if*

$$D(q(x), \phi(x, y), k) = D(p(x), \phi(x, y), k)$$

for all $\phi(x, y) \in L$ *and* $k < \omega$.

It is also natural to consider the foundation rank for forking in simple theories (or even any theory). We call this the SU-rank. Again the main clause is: let $p(x)$ be a complete type, then $SU(p) \geq \alpha + 1$ if p has a forking extension $q(x)$ (also a complete type) such that $SU(q) \geq \alpha$. We have:

REMARK 2.12. Let T be any theory. Then T is supersimple if and only if $SU(p) < \infty$ (namely is ordinal-valued) for all complete types p.

Finally in this section we state how the definability/multiplicity theory aspect of stability can be recovered within the context of simplicity. (See [28].) Definability properties of the ranks $D(-, \phi, k)$ play an important role here.

PROPOSITION 2.13. *Suppose T is simple. Let $p(x) \in S(A)$. The following are equivalent*:

(i) *for some cardinal λ (strictly less than the cardinality of $\mathbf{C}$), for every $B \supset A$, $p(x)$ has at most λ nonforking extensions over B,*
(ii) *every complete extension of $p(x)$ over* $\mathrm{acl}^{\mathrm{eq}}(A)$ *is stationary,*
(iii) *every nonforking extension of $p(x)$ over a model, is definable.*

§3. Examples.

EXAMPLE 3.1. Any stable theory is simple.

EXAMPLE 3.2. The theory of the random graph is the model companion of the theory of a single irreflexive, symmetric binary relation R. It is axiomatized by saying: for all distinct $x_1, \dots, x_n$, $y_1, \dots, y_n$ there is z such that $R(z, x_i)$ for $i = 1, \dots, n$ and $\neg R(z, y_i)$ for all $i = 1, \dots, n$. This theory is ω-categorical, with quantifier-elimination. It is then an easy exercise to show directly that any complete 1-type does not divide over a finite set (in fact over the empty set if p is not algebraic). By Fact 2.4, T is simple, and in fact "$x = x$" has SU-rank 1.

EXAMPLE 3.3. By a pseudofinite field we mean an infinite model of the theory of finite fields. The theory of finite fields was proved to be decidable by James Ax [1]. Pseudofinite fields were shown to be unstable by Duret ([9]). More recently Chatzidakis, Macintyre and van den Dries continued Ax's work, studying definable sets in more detail ([6]). They developed some dimension-theoretic and measure-theoretic properties of definable sets, which we describe briefly. Let F be a pseudofinite field. To any nonempty definable set $X \subset F^n$ can be assigned a dimension $dim(X)$ and a measure $\mu(X)$, the first being a natural number, and the second a positive rational number. $dim(X) = 0$ iff X is finite, in which case $\mu(X)$ is the cardinality of X. Let $\phi(\bar{x}, \bar{y})$ be an L- formula (where L is the language of rings, of course). By an instance of ϕ we mean a set defined by $\phi(\bar{x}, \bar{b})$ for some tuple $\bar{b}$. They pointed out that the set of $(\dim(X), \mu(X))$ as X ranges over instances of ϕ

is finite, and that moreover the condition that $(\dim(X), \mu(X)) = (d, m)$ is a first order condition in the parameters $\bar{b}$ defining X. $\dim(F^n) = n$ and $\mu(F^n) = 1$. Finally, if X, Y are disjoint definable subsets of F^m each of dimension n, then $\mu(X \cup Y) = \mu(X) + \mu(Y)$.

Applying these facts to definable subsets of F itself (and passing to a saturated model), one sees from Fact 2.3, that $\mathrm{Th}(F)$ must be simple. (This argument was pointed out to us by Lou van den Dries.)

In fact in pseudofinite fields, dimension corresponds to SU-rank. (So $SU(F) = 1$.) This was essentially known to Hrushovski, who in [11] developed a stability-style analysis of pseudofinite fields, which we now see to be a "simple" analysis. The dimension function mentioned above is actually the algebraic-geometrical dimension of the Zariski closure, and so forking corresponds to algebraic dependence. Hrushovski proved the Independence Theorem for types over relatively algebraically closed subfields (after adding a fixed suitable set of constants). Hrushovski in [11] also studies perfect PAC (pseudo-algebraically closed) fields with small Galois group. Simplicity can be deduced from his analysis (although in not such an easy way as for pseudofinite fields.) Chatzidakis shows the simplicity of PAC fields with small Galois group and finite degree of imperfection in [5].

Example 3.4. *$ACFA$ is the model companion of the theory of fields equipped with an automorphism σ. The existence of the model companion, as well as a deep model-theoretic study of $ACFA$ was carried out in [4]. (The existence of the model companion was also noticed in earler unpublished work by van den Dries, Macintyre and Wood.) Let (F, σ) be a saturated model of $ACFA$. For a subset A of F, let $< A >$ denote the smallest subfield of F containing A and closed under σ and its inverse. Chatzidakis and Hrushovski introduced the following notion of independence: B is independent from C over A if $\langle A \cup B \rangle$ is algebraically independent from $\langle A \cup C \rangle$ over $\langle A \rangle$. They proved the Independence Theorem for types $p(x) \in S(A)$ where A equals the field-theoretic algebraic closure of $\langle A \rangle$. It follows that $\mathrm{Th}(F)$ is simple, and that this notion of independence coincides with nonforking (nondividing). Alternatively one can show directly that this notion of independence coincides with nondividing. $\mathrm{Th}(F)$ is supersimple, but there are types with infinite SU-rank (such as the type of an element a such that $\{ \sigma^k(a) : k \in \omega \}$ is algebraically independent).

Examples 3.5. In [5], stable theories with "generic" predicates and/or automorphisms are studied. This means the following. One begins with a complete theory T which has quantifier elimination in a language L. Let P be a new predicate symbol (of any arity). Let $L(P)$ be the language obtained by adjoining P to L. The issue is now whether or not T has a model companion in the language $L(P)$. If it does, we call the model companion T_P. (In fact there is a simple necessary and sufficient condition for T_P to exist: the quantifier "there exist infinitely many" should be eliminable in T.) The basic result is that if T is stable (or just simple), then any completion of T_P is simple. Forking in a model (M, P) of T_P corresponds to forking in the L-reduct M.

One could try to do the same thing for automorphisms in place of predicates. Here we start with stable T, again with quantifier elimination. Let $L(\sigma)$ be obtained by adjoining a unary function symbol σ. T_σ^0 is $T \cup$ "σ is an automorphism". The issue of whether or not T_σ^0 has a model companion T_σ is more complicated (we have no nice necessary and sufficient condition), but if it does, then T_σ is simple. (Of course Example 3.4 is a special case.) Recent work of Hrushovski and Chatzidakis-Wood shows that for T, the theory of differentially closed fields of characteristic 0, T_σ exists. It is at present unknown if T_σ exists for T the theory of separably closed fields of some fixed Ersov invariant.

Example 3.6. The notion of a smoothly approximable (or smoothly approximated) structure was introduced, it seems, by Lachlan. Such an object is, by definition, a countable relational ω-categorical structure M which is the union of an increasing chain of finite "homogeneous substructures" of M. By a "homogeneous substructure" of M we mean a subset A of M, such that for any finite tuples a, b chosen from A, a has the same type as b in M if and only if there is an automorphism of M which fixes A setwise and takes a to b. By [8], ω-categorical, ω-stable structures are smoothly approximable, and this was, we suppose, the justification for introducing the notion. The fine analysis of ω-categorical, ω-stable structures in [8], consisted of, firstly, reducing the problem to that of understanding "one-dimensional" such structures using stability theory, and then using either the classification of finite simple groups (applied to the automorphism group of that structure) or some deep model-theoretic arguments, to completely classify these latter objects. Smooth approximability was a *consequence* of this fine analysis. In [16], a general study of smoothly approximable structures was undertaken, in which the permutation group-theoretic analysis was present from the beginning. The structures were shown to be built up from certain geometries, essentially variants of infinite-dimensional vector spaces over finite fields, equipped with nondegenerate bilinear forms. Hrushovski and Cherlin proved the quasi-finite axiomatizability of smoothly approximable structures, introducing more model-theoretic notions, reminiscent of stability theory into the matter. We await the final paper, although descriptions of the work appear in [12] and [7]. A rank notion, and consequent notion of independence is developed, for which the Independence Theorem over algebraically closed sets is proved. We conclude from section 2, that smoothly approximable structures are simple. In fact the general theory of simple theories described above is able to substantially simplify parts of the

Cherlin-Hrushovski analysis. For example, the fact that a structure coordinatized by simple structures is simple, is already part of the Shelah theory. In any case more details will appear in [10].

§**4. Lascar strong types.** Our original idea had been to try to prove Theorem 2.9 (the Independence Theorem) for *strong types*, namely for $p(x)$ a strong type rather than a complete type over a model. We have, however, been unable to prove this in general. Rather amazingly, a notion developed by Daniel Lascar in his study of the category of models of a complete theory [23], turned out to be exactly what was needed. Recall first some notation: by $\mathrm{Aut}_A(\mathbf{C})$ we mean the group consisting of those automorphisms of $\mathbf{C}$ which fix A pointwise.

Definition 4.1. Let A be a subset of $\mathbf{C}$.

(i) By $\mathrm{Laut}_A(\mathbf{C})$ (the group of Lascar strong automorphisms of $\mathbf{C}$ over A) we mean the subgroup of $\mathrm{Aut}(\mathbf{C})$ generated by

$$\{\,\mathrm{Aut}_M(\mathbf{C}) : M \supset A\,\}.$$

(ii) We say $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ (a and b have the same Lascar strong type over A) if a and b are in the same orbit under $\mathrm{Laut}_A(\mathbf{C})$.

Lascar points out that for *stable theories*, $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ if and only if $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$.

As we shall also see later, for simple theories, the Lascar strong type of a over A can be viewed as a partial type (over $A \cup \{a\}$).

Clearly if M is a model, then $\mathrm{Laut}_M(\mathbf{C})$ is the same as $\mathrm{Aut}_M(\mathbf{C})$, and thus if a and b have the same type over M then they have the same Lascar strong type over M. So the following theorem from [21] (which we call the Independence Theorem for Lascar strong types) generalizes Theorem 2.9.

Theorem 4.2. *Let T be simple. Let $A \subset B$, $A \subset C$, with B independent from C over A. Let b, c be such that* $\mathrm{tp}(b/B)$ *does not fork over A,* $\mathrm{tp}(c/C)$ *does not fork over A, and* $\mathrm{Lstp}(b/A) = \mathrm{Lstp}(c/A)$. *Then there is d such that* $\mathrm{tp}(d/B\cup C)$ *does not fork over A,* $\mathrm{tp}(d/B) = \mathrm{tp}(b/B)$, $\mathrm{tp}(d/C) = \mathrm{tp}(c/C)$ *and* $\mathrm{Lstp}(d/A) = \mathrm{Lstp}(b/A) = \mathrm{Lstp}(c/A)$.

We will call an equivalence relation *bounded* if it has strictly less than κ equivalence classes (where recall κ is the cardinality of the big model $\mathbf{C}$). At this point it is worthwhile assuming κ to be strongly inaccessible. It is not difficult to see that for a, b, n-tuples say, the equivalence relation $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ is the finest bounded equivalence relation on $\mathbf{C}^n$ which is invariant under $\mathrm{Aut}_A(\mathbf{C})$ (we just say A-invariant). In fact "bounded" corresponds here to "of cardinality at most $2^{|T|+|A|}$".

In [23], Lascar defined when a theory T is *G-compact*. The first author has pointed out an equivalent (but more palatable) definition in [19]:

Definition 4.3. The theory T is G-compact, if for any $A \subset \mathbf{C}$, we have:

(i) $\mathrm{Laut}_A(\mathbf{C})$ is a closed subgroup of $\mathrm{Aut}_A(\mathbf{C})$, and
(ii) for any n there is a partial type $\Sigma_n(x, y)$ over A (where x, y are n-tuples of variables) such that for any n-tuples a, b, $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ if and only if $\models \Sigma_n(a, b)$.

There are, we believe, no known examples of theories which are not G-compact. In any case, the first author has proved:

PROPOSITION 4.4. *Any simple theory is G-compact.*

Among the advantages of this proposition is that Lascar strong types can be considered as ordinary partial types, to which the machinery of forking naturally applies. That is to say, for a an n-tuple, by the Lascar strong type of a over A we mean the partial type $\Sigma_n(x, a)$ where $\Sigma_n(x, y)$ is from Definition 4.3 (ii). Replacing a by b with the same Lascar strong type over A, gives the equivalent partial type $\Sigma_n(x, b)$. The Independence Theorem for Lascar strong types can then be rephrased: whenever $p(x)$ is a Lascar strong type over A, B and C contain A and are independent over A, and $q_1(x)$, $q_2(x)$ are complete types over B, C respectively such that $q_i(x) \cup p(x)$ does not fork over A for $i = 1, 2$, *then* $q_1(x) \cup q_2(x) \cup p(x)$ does not fork over A.

An interesting problem is whether Lascar strong types reduce to strong types in simple theories. Another way of saying this is: for any A is the partial type $\Sigma_n(x, y)$ from Definition 4.3, equivalent to $\{\, E(x, y) : E$ is a formula over A defining a finite equivalence relation $\}$. This is not true for arbitrary theories. Poizat gave the following example: Let the structure M consist of the circle S (complex numbers of norm 1), the real line $\mathbf{R}$ equipped with its ordering and names for the rational numbers, and the binary map f from S^2 to $\mathbf{R}$ defined by: $f(x, y) = |x - y|$ ($|.|$ being the usual norm, and $-$ coming from the usual group structure on S). M has predicates P, Q for S, $\mathbf{R}$ respectively. Let N be a saturated model of $\mathrm{Th}(M)$. Then one can verify quite easily that for all a, b in P, $\mathrm{stp}(a/\emptyset) = \mathrm{stp}(b/\emptyset)$. However the partial type $\{\, |x - y| < 1/n : 0 < n \,\}$ defines a bounded equivalence relation, which is exactly the relation of having the same Lascar strong type over $\emptyset$.

In any case, the first author proves ([17] and [19]), using ideas from Lascar ([24]) and Pillay and Poizat ([29]):

PROPOSITION 4.5. *Let T be a countable, simple, theory with only countably many types over $\emptyset$. Then for any set A, and a,* $\mathrm{Lstp}(a/A)$ *is equivalent to* $\mathrm{stp}(a/A)$.

Steve Buechler came up with some nice arguments purporting to prove the equality of Lascar strong types and strong types in simple theories. Although these arguments did not work in full generality, Buechler [2], Shami, and others noticed that they *did* work under the assumption that for every L-formula $\phi(x, y)$, there is a finite bound to $D(x = x, \phi(x, y), k)$ as k varies. Simple theories satisfying this last condition are called "low". So for low simple theories, Lascar strong types coincide with strong types. There is work currently going on to find examples of simple theories which are not low (unpublished work of Kim as well as [3] for example).

All the examples mentioned in section 3 are low. In fact in all those examples there are direct methods for showing that Lascar strong types coincide with strong types, usually by proving the Independence Theorem for types over algebraically closed sets in the sense of M^{eq} (or in the case of pseudofinite fields and $ACFA$, over algebraically closed subfields).

§**5. Definable groups.** An important ingredient of general stability theory, is the theory of *stable groups*, as developed in full generality by Poizat ([31]). Analogues of this theory for certain, later seen to be simple, theories, such as pseudofinite fields, or existentially closed difference fields, were worked out in [14], [15], [4], [13]. (The analysis in [15] has interesting consequences for arithmetic groups (strong approximation), and the one in [13] has far-reaching consequences for diophantine questions.) In any case, the main point of Poizat's work was to show the interaction between forking and the presence of a transitive group action, in the context of stable theories. Key aspects of the theory were generic types and stabilizers. The second author, in [28] managed to generalize Poizat's theory to the context of simple theories. The analysis was heavily influenced by some unpublished work of Hrushovski on groups of finite S_1-rank (parts of which appear in [14] and [15]). We describe the results.

In the rest of this section T will be a simple theory, and we work as before in the big model $\mathbf{C}$ of T. By a type-definable (or with some ambiguity, ∞-definable) group, we mean a group G whose domain and group operation are both given by partial types. If these partial types are over A, we say that G is type-definable over A. Let us assume now that G is type-definable over $\emptyset$. For a, b in G, $a \cdot b$ denotes the product of a and b in G. Unless we say otherwise partial types $\Sigma(x)$ and complete types $p(x)$ which we talk about will be assumed to imply "$x \in G$". If $\Sigma(x)$ is such a partial type, and $b \in G$, then by $b \cdot \Sigma(x)$ we mean the partial type $\{ \phi(b^{-1}.x) : \phi(x) \in \Sigma(x) \}$.

Definition 5.1. The partial type $\Sigma(x)$ is said to be generic (for G) if for all $a \in G$, the partial type $a \cdot \Sigma(x)$ does not fork over $\emptyset$.

The basic results on generic types are summarized in the next proposition.

Proposition 5.2.

(i) *For any set A, there is a complete type $p(x) \in S(A)$ which is generic for G.*

(ii) *Let $q(x) \in S(A)$. Then q is generic for G if whenever $b \in G$ and a is a realization of q which is independent from b over A, then* $\mathrm{tp}(b.a/A \cup b)$ *does not fork over $\emptyset$.*

(iii) *Let* $p(x) \in S(A)$, $A \subset B$, *and* $q(x) \in S(B)$ *a nonforking extension of* $p(x)$. *Then* $p(x)$ *is generic if and only if* $q(x)$ *is generic.*
(iv) $\mathrm{tp}(a/A)$ *is generic if and only if* $\mathrm{tp}(a^{-1}/A)$ *is generic.*

In fact our definition of generic types was really of *left* generic types. But (iv) of the above proposition shows that the notion is equivalent to right generics. We may notationally confuse a partial type with its set of realizations (so we can talk of a type-definable subset X of G being generic).

We now pass to stabilizers.

DEFINITION 5.3. Let $p(x) \in S(A)$.
(i) By $S(p)$ (the first approximation to the stabilizer of p) we mean
$$\{ a \in G : p(x) \cup a.p(x) \text{ does not fork over } A \}.$$
(ii) By $\mathrm{Stab}(p)$ (the stabilizer of p) we mean the subgroup of G generated by $S(p)$.

REMARK 5.4. Making use of the ranks $D(-, \phi, k)$ one can show that $S(p)$ is the solution set of a partial type over A.

PROPOSITION 5.5. *Let* $p(x) \in S(M)$. *Then* $\mathrm{Stab}(p) = S(p) \cdot S(p)$ (*that is,* $\mathrm{Stab}(p) = \{ a \cdot b : a, b \in S(p) \}$).

The proof of the above proposition makes heavy use of the Independence Theorem. It shows that $\mathrm{Stab}(p)$ is a type definable group.

The connection between generic types and stabilizers is given by:

PROPOSITION 5.6. *Let* $p(x) \in S(M)$. *Then* p *is a generic in* G *if and only if* $\mathrm{Stab}(p)$ *is a subgroup of bounded index in* G.

Frank Wagner [34] has recently done some more work on groups definable in simple theories. This includes chain conditions, a generalization of the Berline-Lascar theory of superstable groups to the supersimple context, as well as the result that any type-definable group in a supersimple theory is an intersection of definable groups.

§6. Canonical bases and hyperimaginaries. The notion of canonical base is one of most subtle, but also powerful, concepts in stability theory. Canonical bases were discovered/invented by Shelah [33]. They play a fundamental role in showing how superstable structures are "coordinatized" by regular types, or, in the finite rank case, by rank one types, and are crucial for geometric stability theory (the study of the fine structure of models of stable theories). Canonical bases in stable theories yield, for a *stationary* type $p(x) \in S(A)$ a "canonical" set of elements of $\mathbf{C}^{\mathrm{eq}}$, which can be viewed as a kind of "field of moduli" for p. Let us assume T to be stable for now. First we want to associate to p a more canonical object, which eliminates the arbitrariness of the set A. A natural candidate is $\hat{p}(x)$ the unique nonforking extension of $p(x)$ in $S(\mathbf{C})$. Another candidate is $X(p)$, the set of stationary types $q(x)$ over small subsets of $\mathbf{C}$ such that $q(x)$ and $p(x)$ have a common nonforking extension. Equivalently, $X(p)$ is the set of stationary types $q(x)$ which are restrictions of $\hat{p}$. It is rather clear that if f is an automorphism of $\mathbf{C}$ then f fixes $\hat{p}$ if and only if f fixes $X(p)$ setwise. So $\hat{p}$ and $X(p)$ are somehow equivalent objects. The second job is to find some set of *elements* which is equivalent to $\hat{p}$ (or $X(p)$). This is quite easy, although one has to pass to $\mathbf{C}^{\mathrm{eq}}$. Recall from section 2, that $\hat{p}$ is *definable*. That is, for each formula $\phi(x, y)$ of L there is a formula $\psi_\phi(y, c_\phi)$ (for some $c_\phi \in \mathbf{C}$) such that $\psi_\phi(y, c_\phi)$ defines the set of those $b \in \mathbf{C}$ such that $\phi(x, b) \in \hat{p}$. Now each such formula $\psi_\phi(y, c_\phi)$ can be considered as an *element* of $\mathbf{C}^{\mathrm{eq}}$, namely as c_ϕ/E_ϕ where $E_\phi(w, z)$ is the definable equivalence relation $\forall x\ (\psi_\phi(x, w) \leftrightarrow \psi_\phi(x, z))$. We define $\mathrm{Cb}(p)$, the canonical base of p, to be the definable closure (in $\mathbf{C}^{\mathrm{eq}}$) of the set of all such elements c_ϕ/E_ϕ as ϕ ranges over L. That is, $\mathrm{Cb}(p)$ is the definable closure of the set of definitions of the global nonforking extension of p. We see that an automorphism f of $\mathbf{C}$ fixes $\hat{p}$ if and only if it fixes $\mathrm{Cb}(p)$ pointwise. Before stating key properties of canonical bases in stable theories, let us remark that for a a tuple from $\mathbf{C}$ and A some subset of $\mathbf{C}^{\mathrm{eq}}$, $\mathrm{tp}(a/A)$ makes sense, the obvious thing to do being to read this in the structure $\mathbf{C}^{\mathrm{eq}}$.

FACT 6.1. Let T be stable. Let $p(x) \in S(A)$ be stationary. Then $\mathrm{Cb}(p)$ is the (unique) smallest definably closed subset A_0 of $\mathrm{dcl}^{\mathrm{eq}}(A)$ such that (i) $p(x)$ does not fork over A_0, and (ii) the restriction of $p(x)$ to A_0 is stationary.

The reader should see quite clearly that the definability part of stability theory plays a key role in the existence of canonical bases in stable theories. So finding a good generalization to simple theories would appear to be problematic. In fact, it can be done, with a little twist. This is recent work of the authors and Bradd Hart [10]. Let us assume T to be simple. We will call a complete type $p(x) \in S(A)$ an *amalgamation base* if the Independence Theorem holds for $p(x)$. That is, whenever B, C are supersets of A which are independent over A, and $q_1(x)$, $q_2(x)$ are nonforking extensions of $p(x)$ over B, C, respectively, then q_1 and q_2 have a common nonforking extension $r(x) \in S(B \cup C)$. So, considering Fact 6.1 above, one would like to show that given an amalgamation base $p(x) \in S(A)$, there is a unique smallest subset A_0 of $\mathrm{dcl}^{\mathrm{eq}}(A)$ such that p does not fork over A_0 and the restriction of p to A_0 is an amalgamation base. We will do this, except, to our current knowledge, we have to pass not only to imaginary elements, but to hyperimaginary elements.

DEFINITION 6.2. By a hyperimaginary element, we mean something of the form $\bar{a}/E$ where $\bar{a}$ is a tuple from $\mathbf{C}$ of possibly infinite length α (but α is assumed to have cardinality strictly less than the cardinality of $\mathbf{C}$), and

E is an equivalence relation on tuples of length α defined by a partial type $\Phi(\bar{x}, \bar{y})$ over $\emptyset$.

We make a few remarks. An equivalence relation E as in the definition will be called type-definable over $\emptyset$. One could also consider a type-definable equivalence relation E over a set A, but standard methods allow us to identify any E-class with an E'-class for some E' type-definable over $\emptyset$. Note that any tuple (of small length) can be considered as a single hyperimaginary. It does not really make sense to adjoin hyperimaginaries as new sorts, and then treat the resulting structure as a first order one. But one can nevertheless work with hyperimaginaries, remaining completely in the first order context. Suppose for example that $c = \bar{a}/E$ is a hyperimaginary. Let b be an ordinary finite tuple from $\mathbf{C}$. Then one make sense of $\mathrm{tp}(b/c)$ as follows: Let $\phi(x, \bar{y})$ be an ordinary first order L-formula (where length $\bar{y}$ = length $\bar{a}$, but of course only finitely many variables from the sequence $\bar{y}$ actually appear in ϕ). Then the expression $\exists \bar{y}\ (E(\bar{y}, \bar{a}) \wedge \phi(x, \bar{y}))$ is clearly equivalent (by compactness) to a collection of first order formulas in the free variable x with parameters from $\bar{a}$. $\mathrm{tp}(b/c)$ will then be the set of all such expressions, which are true of b, as ϕ ranges over L, and will in particular be a partial type. (One can also make sense of $\mathrm{tp}(b/c)$ when b is itself a hyperimaginary, as a partial type.) Again it makes sense to talk of a partial type $\Sigma(x)$ dividing or forking over a hyperimaginary element c (by considering automorphisms which fix c for example). Thus, for $C \subset D$, sets of hyperimaginaries, and $p(x)$ a complete type over C, it makes sense to talk of a complete type $q(x)$ over D being a nonforking extension of $p(x)$. A hyperimaginary c will be said to in the definable closure of a set A (possibly also consisting of hyperimaginaries), if c is fixed by any automorphism of $\mathbf{C}$ which fixes A pointwise. We also say that $c \in \mathrm{bdd}(A)$ if c has only boundedly many images under automorphisms which fix A pointwise. Then:

THEOREM 6.3. *Let T be simple. Let $p(x) \in S(A)$ be an amalgamation base (for example A could be a model). Then there is some hyperimaginary c such that*

(i) $c \in \mathrm{dcl}(A)$,
(ii) *$p(x)$ does not fork over c,*
(iii) *the restriction of $p(x)$ to c is an amalgamation base,*
(iv) *whenever $A_0 \subset \mathrm{dcl}(A)$ is a set of hyperimaginaries such that p does not fork over A_0 and the restriction of p to A_0 is an amalgamation base, then $c \in \mathrm{dcl}(A_0)$.*
(v) *whenever $A_0 \subset \mathrm{dcl}(A)$ is a set of hyperimaginaries, and p does not fork over A_0, then $c \in \mathrm{bdd}(A_0)$.*

If c is as in the theorem, we call c (or rather $\mathrm{dcl}(c)$) the canonical base of p, $\mathrm{Cb}(p)$. An important question is whether for low simple theories $\mathrm{Cb}(p)$ can be realized as a set of ordinary imaginary elements. In general one can show that canonical bases are the most complicated kind of hyperimaginaries one could have in simple theories. So for example, if all canonical bases exist as tuples of ordinary imaginaries, then it can be shown that every hyperimaginary is "interdefinable" with a tuple of ordinary imaginaries ([25])

In the stable case, we pointed out that $\mathrm{Cb}(p)$ is a kind of code for the set of stationary types which have a common nonforking extension with p. What is the corresponding family in the simple case? Let Q be the collection of types $q(x) \in S(A)$ which are amalgamation bases (A varying). Let R_0 be the binary relation on Q: q_1 and q_2 have a common nonforking extension. Let R be the equivalence relation on Q generated by R_0. Then $\mathrm{Cb}(p)$ has the feature that an automorphism will fix $\mathrm{Cb}(p)$ pointwise if and only if it fixes the R-class of p. The proof of the existence of $\mathrm{Cb}(p)$ is roughly as follows. Assume $p(x)$ is a complete type over a model M. Let $\bar{m}$ be an enumeration of M, and write p as $p_{\bar{m}}(x)$. Let $r(\bar{y}) = \mathrm{tp}(\bar{m}/\emptyset)$. Let E be the equivalence relation on the set of realizations of r defined by: $\bar{m}_1\ E\ \bar{m}_2$ if and only if $p_{\bar{m}_1}\ R\ p_{\bar{m}_2}$. The main work lies in showing that E is type-definable. The hyperimaginary $\bar{m}/E$ serves as $\mathrm{Cb}(p)$.

In all the examples in section 3, canonical bases exist as subsets on M^{eq}. One can deduce that in those examples any hyperimaginary is equivalent to a (possibly infinite) tuple of imaginaries. In cases such as pseudofinite fields, canonical bases exist as subfields, and one deduces full elimination of imaginaries.

§7. Problems.

Problem 7.1. Describe ω-categorical supersimple structures M. In this context it is quite easy to see, using Theorem 6.3, that canonical bases of types exist in M^{eq} (one need not pass all the way to hyperimaginaries), and so well-known methods allow one to show that definable sets of finite SU-rank are built up from Grassmannians over definable sets of SU-rank 1. The context here is definitely more general than that of Example 3.6. For example, the random graph is simple of SU-rank 1, but is not smoothly approximable. An obvious conjecture (parallelling the situation for ω-categorical superstable structures) is that ω-categorical supersimple structures have finite SU-rank, and are modular. Modularity refers here to the geometric behavior of forking, and says that, in M^{eq}, any algebraically closed sets A and B are independent over their intersection. However we have recently heard that Hrushovski has constructed counterexamples to this conjecture. So it seems that an ω-categorical SU-rank 1 set may have non modular geometry. This raises the issue of trying to see how complicated such a geometry could be.

Problem 7.2. Investigate the connection between pseudofiniteness and simplicity. We call an L-structure M pseudofinite if every L-sentence true in M is true in a finite L-structure. We make here the rather outrageous conjecture that if M is ω-categorical and pseudofinite, then $\mathrm{Th}(M)$ is simple.

PROBLEM 7.3. Let F be an infinite field whose theory is supersimple. Must F be *PAC*? Recall that the field F is said to be *PAC* if every absolutely irreducible variety over F has an F-rational point. Earlier work of Poizat and the second author ([30]) implies that if $\mathrm{Th}(F)$ is supersimple, then F is perfect, and has only finitely many extensions of any given degree. The general problem reduces to the case of curves: namely given a saturated supersimple field F and an irreducible curve V defined over F, we want to find an F-rational generic point of V. Chatzidakis and the third author have shown this in the cases where V is a rational curve and where V is a generic hyperelliptic curve (in characteristic 0). This problem is quite pleasant in that one sees that in the same way as some elementary Galois theory was involved in the proof (by Macintyre and in full generality Cherlin-Shelah) that a superstable fields is algebraically closed, the new problem will require some possibly nontrivial input from the theory of curves.

PROBLEM 7.4. Do Lascar strong types coincide with strong types in simple theories?

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