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ANALYTIC IDEALS

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§1. Introduction. Ideals and filters of subsets of natural numbers have been studied by set theorists and topologists for a long time. There is a vast literature concerning various kinds of ultrafilters (or, dually, maximal ideals). There is also a substantial interest in nicely definable (Borel, analytic) ideals—these by old results of Sierpiński are very far from being maximal—and the structure of such ideals will concern us in this announcement. In addition to being interesting in their own right, Borel and analytic ideals occur naturally in the investigations of compact subsets of the space of all Baire class 1 functions on a Polish space (Rosenthal compacta), see [12, 18]. Also, certain objects associated with such ideals are of considerable interest and were quite extensively studied by several authors. Let us list here three examples; in all three of them I stands for an analytic or Borel ideal.

1. The partial order induced by I on $\mathcal{P}(\omega)$: $X \leq_I Y$ iff $X \setminus Y \in I$ ([16]) and the partial order $(I, \subset)$ ([18]).
2. Boolean algebras of the form $\mathcal{P}(\omega)/I$ and their automorphisms ([6, 5, 19, 20]).
3. The equivalence relation associated with I: $XE_I Y$ iff $X \triangle Y \in I$ ([4, 14, 15, 9]).

In Section 4, we will have an opportunity to state some consequences of our results for equivalence relations as in 3.

From this point on we identify $\mathcal{P}(\omega)$ in the standard fashion with $2^\omega = \{0, 1\}^\omega$ and treat it as a topological space (with the product topology on 2^ω where $2 = \{0, 1\}$ is given the discrete topology).

Three natural classes of ideals play a particularly important role: ideals induced, in the manner explained below, by lower semicontinuous submeasures, Polishable ideals, and p-ideals. It is one of the main consequences of our results that these three classes coincide in the realm of analytic ideals.

We explain first the relation between submeasures and ideals on ω. We call $\phi : 2^\omega \to [0, \infty]$ a *submeasure on* ω if $\phi(X) \leq \phi(Y)$ whenever $X \subset Y \in 2^\omega$, $\phi(X \cup Y) \leq \phi(X) + \phi(Y)$ for any $X, Y \in 2^\omega$, and $\phi(\{n\}) < \infty$ for any $n \in \omega$. A submeasure on ω is called *lower semicontinuous* if it is lower semicontinuous as a function from 2^ω regarded with the product topology,

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that is, if $X_n \to X$ implies $\liminf_n \phi(X_n) \geq \phi(X)$ or, equivalently (for submeasures), if $\phi(X) = \lim_n \phi(X \cap n)$ for all $X \in 2^\omega$. We associate with a lower semicontinuous submeasure ϕ two ideals on ω. The first one called the *exhaustive ideal of* ϕ and the second one the *finite ideal of* ϕ.

$$\mathrm{Exh}(\phi) = \{X \in 2^\omega : \phi(X \setminus m) \to 0 \text{ as } m \to \infty\}$$

$$\mathrm{Fin}(\phi) = \{X \in 2^\omega : \phi(X) < \infty\}.$$

It is obvious that $\mathrm{Exh}(\phi) \subset \mathrm{Fin}(\phi)$. A lower semicontinuous submeasure is called *finite* if $\phi(\omega) < \infty$, that is, $\mathrm{Fin}(\phi) = 2^\omega$. A lower semicontinuous submeasure on ω is called *exhaustive* if $\phi(X) < \infty$ implies $\phi(X \setminus m) \to 0$, that is, $\mathrm{Exh}(\phi) = \mathrm{Fin}(\phi)$. It is easy to see that a lsc submeasure ϕ is exhaustive iff for any family X_n, $n \in \omega$, of pairwise disjoint subsets of ω with $\phi(\bigcup_n X_n) < \infty$ we have $\phi(X_n) \to 0$. This shows that our definition agrees with what is usually called an exhaustive submeasure. As was pointed out to me by A. S. Kechris ideals of the form $\mathrm{Fin}(\phi)$ are implicit in K. Mazur's paper [16].

An ideal on ω will be sometimes regarded as a group with symmetric difference as the group operation. An ideal I on ω is called *Polishable* if there exists a Polish group topology τ on I such that the family of Borel sets with respect to τ is equal to the family of Borel subsets of I with respect to the topology inherited from 2^ω. This class of ideals was first studied by Kechris and Louveau in [9].

An ideal I on ω is called a *p-ideal* if for any sequence $X_n \in I$, $n \in \omega$, there exists $X \in I$ such that $X_n \setminus X$ is finite for all n. Analytic p-ideals were studied by Todorcevic in [18].

The following notion of reducing one ideal to another will play a crucial role in the sequel. If I and J are two ideals on ω, we write $J \leq_f I$ if there exists a finite-to-one function $h : \omega \to \omega$ such that $X \in J$ iff $h^{-1}(X) \in I$. So, $J \leq_f I$ means that J is below I with respect to the Rudin-Keisler order and the function witnessing it is finite-to-one.

I would like to finish this section with giving several examples of Borel ideals that will reoccur later on:

$$\mathrm{Fin} = \{X \in 2^\omega : X \text{ is finite}\}$$
$$\mathrm{Fin}^\omega = \{X \in 2^{\omega\times\omega} : \forall n \in \omega\ (\{n\} \times \omega) \cap X \text{ is finite}\}$$
$$I_1 = \{X \in 2^{\omega\times\omega} : \exists n\ X \subset n \times \omega\}.$$

(In some of these examples, as well as in some others later in the paper, the ideal being defined does not consists of subsets of ω but rather of subsets of another countable set. In this case, one can think of this countable set as being identified with ω via a fixed bijection or else one can generalize all the definitions and results in the text to ideals defined on arbitrary countable sets.)

§2. A dichotomy for analytic ideals.

THEOREM 2.1. *Let I be an analytic ideal on ω. Then either $I_1 \leq_f I$ or there exists a finite, lower semicontinuous submeasure ϕ on ω such that $I = \mathrm{Exh}(\phi)$.*

Here are some comments on the proof of this theorem. $I = \mathrm{Exh}(\phi) \Rightarrow I_1 \not\leq_f I$ follows from a theorem of Kechris and Louveau [9]. For an outline of a direct argument the reader can consult the proof of Theorem 3.1. Below we outline the proof of $I_1 \not\leq_f I \Rightarrow I = \mathrm{Exh}(\phi)$.

Each ideal of the form $\mathrm{Exh}(\phi)$ carries a natural Polish topology (see Theorem 3.1, (iii) $\Rightarrow$(i)), so on each such ideal we can define a notion of smallness, namely, small sets are subsets of $\mathrm{Exh}(\phi)$ which are meager with respect to this Polish topology. If the dichotomy "$I_1 \leq_f I$ or $I = \mathrm{Exh}(\phi)$" is to be true, one may hope to define a notion of "small" sets on all ideals and then in the case when $I_1 \not\leq_f I$ prove that these sets behave similarly to meager sets: for instance, are closed under taking countable unions. The following notion proved to be the right one.

A set $A \subset 2^\omega$ is called *small* if there exists $X \in I$ such that $\{Y \cap X : Y \in A\}$ is meager in 2^X.

The first important step in the proof is a justification of the following technical dichotomy.

Let I be an analytic ideal. Then precisely one of the following two possibilities holds.

(i) *$I_1 \leq_f I$;*
(ii) *countable unions of small sets are small.*

We now define a filter which is in a sense dual to I. It is not, however, a filter of subsets of ω but a filter of hereditary compact subsets of 2^ω. Let

$$C(I) = \{K \subset 2^\omega : K \text{ hereditary, compact, and such that } \forall X \in I \exists n \in \omega\ X \setminus n \in K\}.$$

What ties up $C(I)$ with the technical dichotomy above is the fact that, if $K \subset 2^\omega$ is hereditary and compact, then $K \in C(I)$ if, and only if, K is not small. The second important step in the proof consists of constructing a sequence (K_n) of elements of $C(I)$ which is cofinal in $C(I)$ with respect to reverse inclusion and has the additional property that, for all n, $\{X \cup Y : X, Y \in K_{n+1}\} \subset K_n$. To produce such a sequence we use, among other things, the technical dichotomy stated above as well as some results of Kechris-Louveau-Woodin [10] and Talagrand [17]. Moreover, it turns out that I is completely determined by (K_n), namely, $X \in I$ iff $\forall n \exists m\ X \setminus m \in K_n$.

Having produced this sequence, we finish off the proof as follows. First of all we can assume that $K_0 = 2^\omega$ and (by taking every other element of the original sequence (K_n)) that $\{X \cup Y \cup Z : X, Y, Z \in K_{n+1}\} \subset K_n$. Now, we make the following definitions which are motivated by the standard proof of the Birkhoff-Kakutani metrization theorem for groups with a countable basis at the identity. (The connection with topological group theory will become clearer in the next section.) For $X \in [\omega]^{<\omega}$, put

$$\psi_1(X) = \inf\{2^{-n} : X \in K_n\} \text{ and}$$

$$\psi_2(X) = \inf\left\{\sum_{i=0}^{m-1} \psi_1(X_i) : X \subset \bigcup_{i<m} X_i,\ X_i \in [\omega]^{<\omega},\ m > 1\right\}.$$

Now define for $X \in 2^\omega$

$$\phi(X) = \sup_{l\in\omega} \psi_2(X \cap l).$$

It turns out, with some amount of work, that ϕ is a lower semicontinuous submeasure on ω and that $\mathrm{Exh}(\phi) = \{X \in 2^\omega : \forall n \exists m\ X \setminus m \in K_n\}$, but this last set is simply I.

§3. Structure of ideals of the form Exh(ϕ). The next theorem supplies an internal, structural characterization of ideals of the form $\mathrm{Exh}(\phi)$: they are precisely the analytic p-ideals. Moreover, it also turns out that such ideals are exactly those I which carry a Polish topology which agrees with the Borel and the algebraic structures on I.

THEOREM 3.1. *Let I be an ideal on ω. Then the following conditions are equivalent.*

(i) *I is Polishable.*
(ii) *I is an analytic p-ideal.*
(iii) *There is a finite, lower semicontinuous submeasure ϕ on ω with $I = \mathrm{Exh}(\phi)$.*

We outline here the proof of Theorem 3.1. One shows that (iii) implies both (i) and (ii) and that (i) and (ii) imply that $I_1 \not\leq_f I$ which finishes the proof by Theorem 2.1.

(iii) ⇒(i) If ϕ be a lower semicontinuous submeasure with $I = \mathrm{Exh}(\phi)$, let

$$d(X, Y) = \phi(X \triangle Y) \text{ for } X, Y \in I.$$

It turns out that d is an invariant, complete, separable metric on I.

(iii) ⇒(ii) If ϕ is a finite, lower semicontinuous submeasure, then it is not difficult to see that $\mathrm{Exh}(\phi)$ is $F_{\sigma\delta}$, so analytic. That $\mathrm{Exh}(\phi)$ is a p-ideal is shown by a direct calculation.

(i) $\Rightarrow I_1 \not\leq_f I$. This is really a consequence of a theorem due to Kechris and Louveau [9], but can also be shown directly: if I were Polishable and $I_1 \leq_f I$, then it would follow that I_1 is Polishable, which is refuted by a Baire category argument.

(ii) $\Rightarrow I_1 \not\leq_f I$. Again, if I were a p-ideal and $I_1 \leq_f I$, then I_1 would itself be a p-ideal. However, the sequence $\{n\} \times \omega$, $n \in \omega$, of elements of I_1 is not almost included in any element of I_1, so we would get a contradiction.

The following corollary will turn out to be of interest in Section 4.

COROLLARY 3.2. *Let I be an ideal on ω. Then I is locally compact Polishable iff there is $A \subset \omega$ such that $I = \{X : X \cap A$ is finite$\}$.*

The next theorem contains a dichotomy for analytic p-ideals.

THEOREM 3.3. *Let I be an analytic p-ideal on ω. Then precisely one of the following possibilities holds.*

(i) $\mathrm{Fin}^{\omega} \leq_f I$.

(ii) *There exists a lower semicontinuous, exhaustive submeasure ψ on ω such that $I = \mathrm{Fin}(\psi)(= \mathrm{Exh}(\psi))$.*

The proof of this theorem relies on Theorem 3.1, which guarantees that $I = \mathrm{Exh}(\phi)$ for a lower semicontinuous submeasure ϕ and splits according to the following two cases.

Case 1. $\forall \varepsilon > 0 \exists X\ \phi(X) < \varepsilon$ and $\phi(X \setminus m) \not\to 0$.

Case 2. $\exists \varepsilon > 0 \forall X\ \phi(X) < \varepsilon \Rightarrow \phi(X \setminus m) \to 0$.

The following quite surprising corollary on Borel complexity of analytic p-ideals can be established using Theorems 3.1 and 3.3.

COROLLARY 3.4. *If I is an analytic p-ideal, then it is $F_{\sigma\delta}$. Moreover, it is either $F_{\sigma\delta}$-complete or F_{σ}.*

The first part follows from Theorem 3.1 and the easy observation that ideals of the form $\mathrm{Exh}(\phi)$ are $F_{\sigma\delta}$. The moreover part is a consequence of Theorem 3.3 and the facts that Fin^{ω} is $F_{\sigma\delta}$-complete and that $\mathrm{Fin}(\psi)$ is F_{σ}.

The following picture emerges from Theorems 3.1 and 3.2, Corollary 3.2, and a result of Mazur [16] (below ϕ stands for a lower semicontinuous submeasure):

I is Polishable $\Leftrightarrow I = \mathrm{Exh}(\phi)$ for a finite ϕ;

I is $F_{\sigma} \Leftrightarrow I = \mathrm{Fin}(\phi)$ for a ϕ (this is implicit in [16]);

I is Polishable and $F_{\sigma} \Leftrightarrow I = \mathrm{Exh}(\phi) = \mathrm{Fin}(\phi)$ for an exhaustive ϕ;

I is Polishable with a locally compact topology $\Leftrightarrow I = \{X \subset \omega : |X \cap A| < \aleph_0\}$ for some $A \subset \omega$.

§4. **Applications to equivalence relations.** We will state here some consequences of results from Section 3 concerning the equivalence relations induced by ideals on ω. Recall that if I is an ideal on ω, the equivalence relation E_I induced by I on 2^ω is defined by

$$XE_IY \text{ iff } X \triangle Y \in I.$$

Some of these equivalence relations have their own names used widely in literature; so, we have

$$E_0 = E_{\mathrm{Fin}},\ E_1 = E_{I_1},\ \text{and}\ E_0^\omega = E_{\mathrm{Fin}^\omega}.$$

The equivalence relations induced by ideals are of importance in the study of Borel equivalence relations. (Note that if I is Borel, then E_I is Borel as well.) On the one hand, they provide canonical examples of equivalence relations which occur in various dichotomy theorems. (In dichotomy theorems one proves that an equivalence relation on a Polish space is either simple or very complicated. The complexity of the equivalence relation is usually witnessed by the fact that it embeds, in a suitable sense, a canonical complicated equivalence relation, and these complicated equivalence relations are usually equivalence relations induced by Borel ideals.) So, for instance, E_0 occurs in the Glimm-Effros dichotomy for F_σ equivalence equivalence relations induced by actions of Polish groups as well as extensions of this result due to Kechris-Harrington-Louveau [2] and Hjorth-Solecki [3]; while E_1 occurs in the dichotomy theorem for hypersmooth equivalence relations proved by Kechris and Louveau [9]. On the other hand, the equivalence relations induced by ideals give examples witnessing how rich the class of Borel equivalence relations is (see papers by Louveau-Velickovic [14] and Mazur [15]) and provide the testing ground for various hypotheses conjectured about the structure of general Borel equivalence relations.

If E and F are Borel equivalence relations on Polish spaces X and Y, respectively, we write $E \sqsubseteq_c F$ and say that F *continuously embeds* E if there exists a homeomorphic embedding $f : X \to Y$ such that xEy iff $f(x)Ff(y)$. It is not difficult to notice that if $I \leq_f J$, I, J ideals on ω, then $E_I \sqsubseteq_c E_J$. Corollary 4.1 is therefore a consequence of results in Section 3. To motivate it, recall that it was proved by Kechris and Louveau in [9] that if an equivalence relation is induced by a Polish group action, then it does not continuously embed E_1; so, E_I does not embed E_1 if I is a Polishable ideal. They asked if this is the only obstacle for E_I, I a Borel ideal, to embed E_1. This problem provided the motivation for all the research reported here and is answered in part (i) of the corollary.

Corollary 4.1. *Let I be an analytic ideal on ω. Then*

(i) *either $E_1 \sqsubseteq_c E_I$ or I is Polishable*;

(ii) *if additionally I is Polishable, then $E_0^\omega \sqsubseteq_c E_I$ or E_I is F_σ.*

It is worth mentioning that, as stated above, by [9] we cannot have $E_1 \sqsubseteq_c E_I$ if I is Polishable and also that an F_σ equivalence relation cannot continuously embed E_0^ω since E_0^ω is $F_{\sigma\delta}$ complete. These remarks show that in (i) and (ii) of Corollary 4.1 the two possibilities exclude each other. Let us also mention here that (ii) is not true in the more general context of $F_{\sigma\delta}$ ideals. Van Engelen's paper [1] contains examples of $F_{\sigma\delta}$ ideals which are neither F_σ nor $F_{\sigma\delta}$ complete. Such ideals cannot fulfill either of the conditions of dichotomy (ii) from Corollary 4.1. (I am grateful to A. S. Kechris for bringing van Engelen's result to my attention.)

If E and F are Borel equivalence relations on Polish spaces X, Y, respectively, we say that F *Borel reduces* E, in symbols $E \leq_B F$, if there exists a Borel function $f : X \to Y$ such that xEy iff $f(x)Ff(y)$. It is of considerable interest to find out how complicated the order $\leq_B$ is among Borel equivalence relations or among some subclasses of Borel equivalence relations, particularly, among the class of Borel equivalence relations with countable equivalence classes. After some initial results of Just [4] and Woodin, Louveau and Velickovic in [14] proved that $\leq_B$ among all Borel equivalence relations is very complicated indeed: it embeds the partial order $X \leq^* Y \Leftrightarrow |X \setminus Y| < \aleph_0$ for $X, Y \in 2^\omega$. The equivalence relations used in [14] are equivalence relations induced by $F_{\sigma\delta}$ ideals. Later Mazur in [15] produced a similar phenomenon for equivalence relations induced by F_σ ideals. Kechris then asked if the same can be proved for equivalence relations which are induced by Borel ideals and are reducible to Borel equivalence relations with countable equivalence classes. He pointed out that by [7] the last condition implies that such ideals carry Polish locally compact group topologies which are compatible with their Borel structure. But now from Corollary 3.2 we obtain that such equivalence relations are very simple and the Louveau-Velickovic phenomenon cannot be found among them.

COROLLARY 4.2. *Let I be an analytic ideal. Then E_I is Borel reducible to a Borel equivalence relation with countable equivalence classes if and only if $I = \{X \in 2^\omega : X \cap A$ is finite$\}$ for some $A \subset \omega$.*

Some further results concerning equivalence relations induced by Borel ideals were recently obtained by Kechris [8].

§5. An application to σ-ideals of Borel sets. In [11], we (that is A. S. Kechris and myself) gave an abstract characterization of the σ-ideal of meager sets. The motivation for proving this result came from a question of Kunen [13]: is it true that the only σ-ideals of Borel subsets of 2^ω enjoying certain (specified in [13]) invariance and structural properties are the σ-ideals of meager sets, of Lebesgue measure zero sets, and the intersection of the two? To state the result from [11], we will need a few new notions. A family of Borel sets is F_σ-*supported* if any set from the family is contained in an F_σ from the family. A family $\mathcal{I}$ of Borel subsets of a Polish space X is *analytic* on G_δ sets if for any Polish space Y and any G_δ set $G \subset X \times Y$ we have that $\{y \in Y : \{x \in X : (x, y) \in G\} \in \mathcal{I}\}$ is analytic. For a family $\mathcal{F}$ of closed subsets of a Polish space X, we put

$$\mathrm{MGR}(\mathcal{F}) = \{Z \subset X : Z \text{ Borel and } Z \cap F \text{ is meager in } F \text{ for any } F \in \mathcal{F}\}.$$

THEOREM 5.1 (Kechris-Solecki [11]). *Let $\mathcal{I}$ be a family of Borel subsets of a Polish space. Then the following two conditions are equivalent*

(i) $\mathcal{I} = \mathrm{MGR}(\mathcal{F})$ *for a family $\mathcal{F}$ of closed sets well-ordered by inverse inclusion.*
(ii) $\mathcal{I}$ *is a σ-ideal which is F_σ-supported and analytic on G_δ sets.*

Let me mention here that it is possible to replace the definability condition "analytic on G_δ sets" in (ii) by the structural assumption that I fulfills the countable chain condition (for a precise formulation see [11]). Also, as a straightforward consequence of Theorem 5.1 one obtains that if X is a Polish group and the ideal is assumed to be translation invariant, in addition to the conditions listed in (ii), then it is the ideal of all meager subsets of X.

It would be desirable to have a theorem analogous to 5.1 for σ-ideals induced by Borel probability measures. This should bring us closer to the full solution of Kunen's problem. There is one concession that we have to make from the beginning. Each finite Borel measure is also a Maharam submeasure (for a definition see below) and, on the other hand, Maharam's conjecture says that for any Maharam submeasure μ there is a finite Borel measure ν such that $\mu(A) = 0$ iff $\nu(A) = 0$ for any Borel set A. Maharam's conjecture remaining unresolved, it seems natural to attempt first to characterize the σ-ideals induced by Maharam submeasures. It turns out that the results from Section 3 can be used to give such a characterization.

Let X be a Polish space. A function $\mu :\ \mathrm{Bor}(X) \to [0, \infty)$, X a Polish space, is called a *Maharam submeasure* if

(i) $A \subset B$ implies $\mu(A) \le \mu(B)$;
(ii) $\mu(A \cup B) \le \mu(A) + \mu(B)$;
(iii) if $A_n \to A$ (that is, the characteristic functions of A_n, χ_{A_n}, tend pointwise to χ_A), then $\mu(A_n) \to \mu(A)$.

A family $\mathcal{I}$ of Borel subsets of a Polish space X is called G_δ-*supported* if for any $A \in \mathcal{I}$ there exists a G_δ set $B \in \mathcal{I}$ with $A \subset B$. For a family $\mathcal{I}$ of Borel subsets of 2^ω, let

$$\hat{\mathcal{I}} = \{X \subset 2^{<\omega} : [X] \in \mathcal{I}\}$$

where $[X] = \{x \in 2^\omega : x|k \in X \text{ for infinitely many } k \in \omega\}$. A word of explanation might be in place here. Any G_δ subset of 2^ω is coded in a natural way by a set $X \subset 2^{<\omega}$—the G_δ set coded by X is $[X]$. So, $\hat{\mathcal{I}}$ is the collection of all such codes for G_δ sets from $\mathcal{I}$.

THEOREM 5.2. *Let $\mathcal{I}$ be a family of Borel subsets of 2^ω. Then the following conditions are equivalent.*

(i) $\mathcal{I} = \{A \in Bor(X) :\ \mu(A) = 0\}$ *for a Maharam submeasure μ.*

(ii) *$\hat{\mathcal{I}}$ is a p-ideal and $\mathcal{I}$ is G_δ-supported and analytic on G_δ sets.*

The analogy between Theorems 5.1 and 5.2 seems quite striking especially if one realizes that the condition "$\hat{\mathcal{I}}$ is a p-ideal" is a strengthening of the condition that $\mathcal{I}$ is a σ-ideal. However, I do not know if one can interchange the two conditions, or if one can replace the definability assumption that I be analytic on G_δ sets by a structural assumption on I (as is the case in Theorem 5.1).

We will comment on the proof of the more interesting implication (ii) $\Rightarrow$(i). First, one should notice that $\hat{\mathcal{I}}$ is analytic and so by Theorem 2.1 there exists a lower semicontinuous submeasure $\phi :\ 2^{<\omega} \to [0, \infty)$ such that $\hat{\mathcal{I}} = \mathrm{Exh}(\phi)$. For a Borel set $B \subset 2^\omega$, let

$$\mu(B) = \inf\{\phi(X) :\ X \subset 2^{<\omega} \text{ and } B \subset [X]\}.$$

It turns out that μ is a Maharam submeasure and that $\mathcal{I}$ is the σ-ideal of μ-zero sets. It is quite easy to see that for $A, B \subset 2^\omega$ Borel, we have $\mu(A \cup B) \leq \mu(A) + \mu(B)$ and, if $A \subset B$, $\mu(A) \leq \mu(B)$, and that if $A \in \mathcal{I}$, then $\mu(A) = 0$. What remains to be shown, and this constitutes the main part of the proof, is that if $\mu(A) = 0$ for a Borel set $A \subset 2^\omega$, then $A \in \mathcal{I}$, and that if $A_n \to A$, $A_n \subset 2^\omega$ Borel, then $\mu(A_n) \to \mu(A)$.

REFERENCES

[1] F. VAN ENGELEN, *On Borel ideals*, ***Annals of Pure and Applied Logic***, vol. 70 (1994), pp. 177–203.

[2] L. HARRINGTON, A. S. KECHRIS, and A. LOUVEAU, *A Glimm-Effros dichotomy for Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 3 (1990), pp. 903–928.

[3] G. HJORTH and S. SOLECKI, *Vaught's conjecture and the Glimm-Effros property for Polish transformation groups*, to appear.

[4] W. JUST, *More mutually irreducible ideals*, preprint, July 1990.

[5] ———, *Repercussions of a problem of Erdos and Ulam on density ideals*, ***Canadian Journal of Mathematics***, vol. 42 (1990), pp. 902–914.

[6] W. JUST and A. KRAWCZYK, *On certain Boolean algebras $\mathcal{P}(\omega)/I$*, ***Transactions of the American Mathematical Society***, vol. 285 (1984), pp. 411–429.

[7] A. S. KECHRIS, *Countable sections for locally compact group actions*, ***Ergodic Theory and Dynamical Systems***, vol. 12 (1992), pp. 283–295.

[8] ———, *Rigidity properties of Borel filters on the integers*, handwritten notes, March 1996.

[9] A. S. KECHRIS and A. LOUVEAU, *A classification of hypersmooth equivalence relations*, ***Journal of the American Mathematical Society***, to appear.

[10] A. S. KECHRIS, A. LOUVEAU, and H. WOODIN, *The structure of σ-ideals of compact sets*, ***Transactions of the American Mathematical Society***, vol. 301 (1987), pp. 263–288.

[11] A. S. KECHRIS and S. SOLECKI, *Approximating analytic by Borel sets and definable chain conditions*, ***Israel Journal of Mathematics***, vol. 89 (1995), pp. 343–356.

[12] A. KRAWCZYK, *On Rosenthal compacta and analytic sets*, ***Proceedings of the American Mathematical Society***, vol. 115 (1992), pp. 1095–1100.

[13] K. KUNEN, *Random and Cohen reals*, ***Handbook of set-theoretic topology***, Elsevier Science Publishers, 1984, pp. 887–911.

[14] A. LOUVEAU and B. VELICKOVIC, *A note on Borel equivalence relations*, ***Proceedings of the American Mathematical Society.***, vol. 120 (1994), pp. 255–259.

[15] K. MAZUR, *A modification of Louveau and Velickovic construction for F_σ ideals*, preprint.

[16] ———, *F_σ ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $\mathcal{P}(\omega)/I$*, ***Fundamenta Mathematicae***, vol. 138 (1991), pp. 103–111.

[17] M. TALAGRAND, *Compacts de fonctions mesurables et filtres non mesurables*, ***Studia Mathematica***, vol. 67 (1980), pp. 13–43.

[18] S. TODORCEVIC, *Analytic gaps*, ***Fundamenta Mathematicae***, vol. 150 (1996), pp. 55–56.

[19] B. VELICKOVIC, *Definable automorphisms of $\mathcal{P}(\omega)/fin$*, ***Proceedings of the American Mathematical Society.***, vol. 96 (1986), pp. 130–135.

[20] ———, *OCA and automorphisms of $\mathcal{P}(\omega)/fin$*, ***Topology and its Applications***, vol. 49 (1993), pp. 1–13.

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