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$\mathrm{HOD}^{L(\mathbb{R})}$ IS A CORE MODEL BELOW Θ

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In this paper we shall answer some questions in the set theory of $L(\mathbb{R})$, the universe of all sets constructible from the reals. In order to do so, we shall assume $\mathrm{AD}^{L(\mathbb{R})}$, the hypothesis that all 2-person games of perfect information on ω whose payoff set is in $L(\mathbb{R})$ are determined. This is by now standard practice. ZFC itself decides few questions in the set theory of $L(\mathbb{R})$, and for reasons we cannot discuss here, ZFC + $\mathrm{AD}^{L(\mathbb{R})}$ yields the most interesting "completion" of the ZFC-theory of $L(\mathbb{R})$.

$\mathrm{AD}^{L(\mathbb{R})}$ implies that $L(\mathbb{R})$ satisfies "every wellordered set of reals is countable", so that the axiom of choice fails in $L(\mathbb{R})$. Nevertheless, there is a natural inner model of $L(\mathbb{R})$, namely $\mathrm{HOD}^{L(\mathbb{R})}$, which satisfies ZFC. (HOD is the class of all hereditarily ordinal definable sets, that is, the class of all sets x such that every member of the transitive closure of x is definable over the universe from ordinal parameters (i.e., "OD"). The superscript "$L(\mathbb{R})$" indicates, here and below, that the notion in question is to be interpreted in $L(\mathbb{R})$.) $\mathrm{HOD}^{L(\mathbb{R})}$ is reasonably close to the full $L(\mathbb{R})$, in ways we shall make precise in §1. The most important of the questions we shall answer concern $\mathrm{HOD}^{L(\mathbb{R})}$: what is its first order theory, and in particular, does it satisfy GCH?

These questions first drew attention in the 70's and early 80's. (See [4, p. 223]; also [12, p. 573] for variants involving finer notions of definability.) During that period, descriptive set theorists proved a number of interesting results concerning $\mathrm{HOD}^{L(\mathbb{R})}$, some of which we shall enumerate in §1 below. Nevertheless, the GCH in $\mathrm{HOD}^{L(\mathbb{R})}$, as well as other basic questions of combinatorial set theory lurking beyond it, remained open. Our work explains why: the methods of "neoclassical" descriptive set theory (as represented, for example, by Moschovakis' book [12]) could not yield the key first-order property of $\mathrm{HOD}^{L(\mathbb{R})}$. This property is that, below $\Theta^{L(\mathbb{R})}$ anyway, $\mathrm{HOD}^{L(\mathbb{R})}$ is a *core model*. (Here $\Theta =_{\text{df.}} \sup\{\alpha \mid \exists f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}$.) One needs the recent progress in relating large cardinals and determinacy ([9], [15], [14]) and the advances in core model theory which have accompanied it ([10], [11], [13]) in order to formulate precisely and prove this key fact. In the end, it is

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not $AD^{L(\mathbb{R})}$, but an existence assertion about mice to which it is equivalent, which is most relevant.

It is part of the basic theory of core models that they satisfy GCH, and so indeed $HOD^{L(\mathbb{R})} \models GCH$. More than that, the powerful fine-structure theory of core models, based on condensation and comparison arguments, applies to the study of $HOD^{L(\mathbb{R})}$.

Our work also solves some other problems from the neoclassical period. We shall show that $HOD^{L(\mathbb{R})}$ is, in a loose sense, constructed by iteration from countable mice. From this construction, by well-known methods, we get that if $\kappa < \Theta^{L(\mathbb{R})}$ and $HOD^{L(\mathbb{R})} \models \kappa$ is regular, then exactly one of the following holds: (1) $HOD^{L(\mathbb{R})} \models \kappa$ is measurable, (2) $\mathrm{cof}^{L(\mathbb{R})}(\kappa) = \omega$. In particular, if $\kappa < \Theta^{L(\mathbb{R})}$ is a successor cardinal of $HOD^{L(\mathbb{R})}$, then $\mathrm{cof}^{L(\mathbb{R})}(\kappa) = \omega$ (and so κ is not a cardinal of $L(\mathbb{R})$). Relativizing this to $HOD^{L(\mathbb{R})}(x)$ for $x \in \mathbb{R}$, we get the "boldface GCH": in $L(\mathbb{R})$, every wellordered family of distinct subsets of κ, where $\kappa < \Theta^{L(\mathbb{R})}$, has size $\leq \kappa$. Finally, turning to $\neg(2) \Rightarrow (1)$ of the dichotomy above, and relativizing it to arbitrary $x \in \mathbb{R}$, we can show that in $L(\mathbb{R})$, every regular $\kappa < \Theta^{L(\mathbb{R})}$ is measurable. (Cf. [12, p. 221].) It is perhaps worth noting that these last two corollaries are statements about $L(\mathbb{R})$ which, at least superficially, do not involve $HOD^{L(\mathbb{R})}$ at all.

Some of the results above (e.g., the "all regulars are measurable" result) we only proved from $AD^{L(\mathbb{R})}$ for $\kappa < (\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$. Our proof for $\kappa < \Theta^{L(\mathbb{R})}$ required a slightly stronger hypothesis. H. Woodin later found the proof of the full results quoted above in these cases.

Our work adapts to finer notions of definability. For example, H_{2n+1} is a core model below $\boldsymbol{\delta}^1_{2n+1}$, and so $H_{2n+1} \models GCH$. (Here

$$\boldsymbol{\delta}^1_{2n+1} =_{\text{df.}} \sup\{\alpha \mid \exists f(f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\} \text{ and } \{(x, y) \mid f(x) \leq f(y)\} \text{ is } \boldsymbol{\Delta}^1_{2n+1})\}.$$

The model H_{2n+1} is a fragment of $HOD^{L(\mathbb{R})}$ corresponding, roughly, to Σ^1_{2n+2} definability from ordinals $< \boldsymbol{\delta}^1_{2n+1}$. cf. [12, p. 573].) In the other direction, Woodin has extended our results to models of the form HOD^M, for certain $M \models AD$ which are larger than $L(\mathbb{R})$. There are interesting open questions in both directions. Looking downward: is it true that for all $x \in \mathbb{R}$ of sufficiently large Turing degree, $HOD^{L[x]} \models GCH$? Looking upward: let M be the minimal model of $AD_{\mathbb{R}}$ containing all ordinals and reals; does HOD^M satisfy GCH?

In §1 we outline some older work on $HOD^{L(\mathbb{R})}$. In §2 and §3 we sketch a proof that $HOD^{L(\mathbb{R})}$ is a core model below $\Theta^{L(\mathbb{R})}$. We have endeavored to make this sketch accessible, and even useful, to the reader who has only a loose grasp of the basics of descriptive set theory and core model theory. On the other hand, the reader for whom a mouse is simply a rodent should not bother with §2 and §3.

§1. Older results about HOD$^{L(\mathbb{R})}$. We shall be discussing facts about HOD$^{L(\mathbb{R})}$ which are expressible in the first order language of $L(\mathbb{R})$, and provable in ZFC + AD$^{L(\mathbb{R})}$. For notational convenience, we "live inside $L(\mathbb{R})$" throughout this section; that is, we assume ZF + AD + $V = L(\mathbb{R})$ throughout.

Since every wellordered set of reals is countable, $\mathbb{R} \cap$ HOD is countable; nevertheless, HOD is close to V. One manifestation of this is Vopenka's theorem ([5]), according to which every set of ordinals, and hence every real, is generic over HOD. In fact, many interesting reals are OD, and thus in HOD. Among the OD sets of reals only the countable ones are literally in HOD; however, Woodin has shown that every OD set of reals has an ∞-Borel code in HOD. (See [14]. An ∞-Borel set of reals is one which is constructed from basic open sets via repeated wellordered union and complementation; an ∞-Borel code is a set of ordinals naturally coding such a construction.) In related work, Woodin showed that HOD $= L[P]$ for some $P \subseteq \Theta$.

An ∞-Borel code of A gives an effective way of answering membership questions about A, but no way of producing members of A. For example, $\mathbb{R}\backslash$HOD has an ∞-Borel code in HOD, but no member in HOD. A *Suslin representation* of A gives more information. Let T be a tree on $\omega \times \kappa$ for some ordinal κ; we say T is a Suslin representation of

$$p[T] = \{x \in {}^{\omega}\omega \mid \exists f \in {}^{\omega}\kappa \forall n (x \restriction n, f \restriction n) \in T\}.$$

Notice that for such a tree T, if $p[T] \neq \emptyset$, then $p[T] \cap L[T] \neq \emptyset$. It follows that $\mathbb{R}\backslash$HOD is an OD (in fact $\Pi_1^{L(\mathbb{R})}$) set of reals having no Suslin representation in HOD. The paper [8] shows, however, that every $\Sigma_1^{L(\mathbb{R})}$ set of reals has a Suslin representation in HOD. This can be used to compute the correctness of HOD: any Σ_1 sentence true in $L(\mathbb{R})$ is true in $L(\mathbb{R})^{\text{HOD}}$. (But $L(\mathbb{R})^{\text{HOD}}$ satisfies "There is a wellorder of $\mathbb{R}$", a Σ_1 sentence which is not true in $L(\mathbb{R})$.)

Actually, [8] gives a Suslin representation T for any complete $\Sigma_1^{L(\mathbb{R})}$ set of reals which is the "tree of a $\Sigma_1^{L(\mathbb{R})}$ scale". Such a T has the following useful properties. First, T is on $\omega \times \boldsymbol{\delta}_1^2$, where $\boldsymbol{\delta}_1^2$ is the least ordinal σ such that $L_\sigma(\mathbb{R})$ is a Σ_1 elementary substructure of $L(\mathbb{R})$. (No smaller T is posssible.) Second, for each $x \in p[T]$, let ℓ_x be the minimal witness that $x \in p[T]$; that is, let $\ell_x(n) =$ least α such that $(\exists f \supseteq \ell_x \restriction n^\frown\langle\alpha\rangle)\forall i((x \restriction i, f \restriction i) \in T)$. Then $\forall\alpha < \boldsymbol{\delta}_1^2 \exists x \in p[T](\ell_x(0) = \alpha)$; moreover, the relation "$x, y \in p[T]$ and $\ell_x(0) < \ell_y(0)$" is $\Sigma_1^{L(\mathbb{R})}$.

According to the following theorem of Becker and Kechris, all of HOD below $\boldsymbol{\delta}_1^2$ is coded into any such tree.

Theorem (Becker, Kechris; cf. [2]). *Let T be the tree of a $\Sigma_1^{(\mathbb{R})}$ scale on a complete $\Sigma_1^{L(\mathbb{R})}$ set; then $V_{\delta_1^2}^{HOD} = V_{\delta_1^2}^{L[T]}$.*

The proof we give in §3 that HOD is a core model uses this theorem. We shall construct a core model $M \subseteq$ HOD such that for some tree T of a $\Sigma_1^{L(\mathbb{R})}$ scale on a complete $\Sigma_1^{L(\mathbb{R})}$ set, $T \in M$. The Becker-Kechris theorem then implies $V_{\boldsymbol{\delta}_1^2}^{\mathrm{HOD}} = V_{\boldsymbol{\delta}_1^2}^{M}$, so that HOD is a core model "below $\boldsymbol{\delta}_1^2$".

HOD is also large in that it satisfies certain large cardinal hypotheses. The roots of this are Solovay's method of constructing ultrafilters using AD, and Kunen's theorem that every ultrafilter on an ordinal $\kappa < \Theta$ is OD ([7]). Becker, Martin, Moschovakis, and the author used these ideas to show that HOD has respectably large cardinals. (For example, say that κ is β-strong if there is an elementary $j : V \to M$ with critical point κ such that $V_\beta \subseteq M$. The author showed $\boldsymbol{\delta}_1^2$ is β-strong in HOD, where β is the least measurable cardinal $> \boldsymbol{\delta}_1^2$ in HOD.) Woodin found a far reaching and definitive extension of this work.

THEOREM (Woodin, [14]). *The following are true in HOD, for* $\delta = (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$ *and* $\theta = \Theta^{L(\mathbb{R})}$*,*

(1) $\forall \beta < \theta$ *(δ is β-strong),*
(2) *θ is Woodin.*

The definition of "Woodin cardinal" can be found, for example, in [9].

Like the Becker-Kechris theorem, this result helped motivate our work. Notice that by (1), if HOD is a core model below $\boldsymbol{\delta}_1^2$, then HOD is a core model below Θ.

Becker [3] showed that ω_1^V is the least measurable cardinal of HOD. Woodin (unpublished) showed that $\boldsymbol{\delta}_1^2$ is the least κ which is β-strong in HOD for all $\beta < \Theta$. These results also give some hints as to how to realize HOD as a core model.

Concerning the GCH in HOD, Becker proved the following.

THEOREM (Becker [1]). *Suppose that* $\alpha \leq \omega_1^V$ *or* α *is reliable; then* $HOD \models 2^\alpha = \alpha^+$.

Adopting a variant of Becker's definition, we say α is *reliable* iff there is a tree T on $\omega \times \alpha$ such that $\forall \beta < \alpha \exists x \in p[T](\ell_x(0) = \beta)$, and the relation "$x, y \in p[T]$ and $\ell_x(0) < \ell_y(0)$" has a Suslin representation. Thus $\boldsymbol{\delta}_1^2$ is reliable. If α is reliable, then $\alpha \leq \boldsymbol{\delta}_1^2$; moreover, there is a closed unbounded set of reliable cardinals $\alpha < \boldsymbol{\delta}_1^2$, as well as an unbounded set of unreliable cardinals $\alpha < \boldsymbol{\delta}_1^2$. Woodin (unpublished) has shown that if α is a reliable cardinal, then no $\kappa < \alpha$ is α-strong in HOD. Conversely, our work here shows that if $\alpha < \Theta^{L(\mathbb{R})}$ is a cardinal such that no $\kappa < \alpha$ is α-strong in HOD, then α is reliable. This then characterizes the reliable cardinals in terms of the large cardinal structure of HOD.

The most immediate inspiration for our work was the work of H. Woodin and the author described in §4 of [13]. It is shown there that if M_ω is the minimal iterable proper class model with ω Woodin cardinals, and N is the result of linearly iterating M_ω so as to send its least measurable cardinal out to ω_1^V, then N and HOD agree on sets of rank $\leq \omega_1^V$. This re-proves Becker's result that ω_1^V is the least measurable of HOD, and also re-proves Becker's result on the GCH below ω_1^V. (It is amusing to note that this also re-proves Solovay's theorem that every $A \subseteq \omega_1^V$ is constructible from a real. For $A \in \mathrm{HOD}(x)$ for some real x, and the results of §4 of [13], relativised to x, then show A is constructible from $M_\omega(x)$.) The results of §4 of [13] clearly suggest that the whole of HOD might be an iterate of M_ω, and it was the attempt to show this which led to the present work.

§2. Some core model theory. This section contains a sketchy summary of some of the core model theory we shall need. All lies are white. The reader can find a full and true account in [11] and [13]. He can find a brief history and overview of the subject in [6], or in the introduction to [10].

Let M be a transitive model; then a (κ, ν) *extender over* $\mathcal{M}$ is a system of $\mathcal{M}$-ultrafilters coding an embedding $i_E : \mathcal{M} \to \mathcal{N}$ such that $\kappa = \mathrm{crit}(i_E)$ and $\nu < i_E(\kappa)$. We write $\kappa = \mathrm{crit}(E)$, $\nu = lh(E)$, and $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, E)$. A sequence $\vec{E} = \langle E_\alpha \mid \alpha < \tau\rangle$ of extenders is *coherent* if for all $\alpha < \tau$,

(1) E_α is an extender over $L[\langle E_\beta \mid \beta < \alpha\rangle]$,
(2) $\beta < \alpha \Rightarrow lh(E_\beta) < lh(E_\alpha)$, and
(3) $i_{E_\alpha}(\vec{E}) \restriction \alpha + 1 = \vec{E} \restriction \alpha$.

One can think of a coherent sequence as one in which the extenders are listed in order of increasing strength, without leaving gaps. A *premouse* is a structure $\mathcal{M} = (J_\alpha^{\vec{E}}, \in, \vec{E})$, where $\vec{E}$ is coherent; we call $\vec{E}$ the $\mathcal{M}$-sequence, and for $\beta < \alpha$ we set $\mathcal{J}_\beta^{\mathcal{M}} = (J_\beta^{\vec{E}\restriction\gamma}, \in, \vec{E} \restriction \gamma)$, where $\gamma = \{\eta \mid lh(E_\eta) \leq \omega\beta\}$.

Let $\mathcal{M}$ be a premouse and τ an ordinal. The *full iteration game on* $\mathcal{M}$ *of length* τ, or $\mathcal{G}(\mathcal{M}, \tau)$, is played as follows. Players I and II cooperatively build an iteration tree $\mathcal{T}$ on $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$. Player I moves at successor stages $\alpha + 1 < \tau$ by choosing an extender $E_\alpha^{\mathcal{T}}$ from the $\mathcal{M}_\alpha^{\mathcal{T}}$-sequence and a $\beta \leq \alpha$ such that

$$\forall\gamma(\beta \leq \gamma < \alpha \Rightarrow \mathrm{crit}(E_\alpha^{\mathcal{T}}) < lh(E_\gamma^{\mathcal{T}})).$$

We appoint β the $\mathcal{T}$ predecessor of $\alpha + 1$, and set

$$\begin{aligned}\mathcal{M}_{\alpha+1}^{\mathcal{T}} \;=\; & \mathrm{Ult}(\mathcal{P}, E_\alpha^{\mathcal{T}}),\ \text{where } \mathcal{P} = \mathcal{J}_\gamma^{\mathcal{M}_\beta^{\mathcal{T}}} \text{ for the largest } \gamma \\ & \text{such that } E_\alpha^{\mathcal{T}} \text{ is an extender over } \mathcal{J}_\gamma^{\mathcal{M}_\beta^{\mathcal{T}}}.\end{aligned}$$

Player I wins, and the game ends, if $\mathcal{M}_{\alpha+1}^{\mathcal{T}}$ is illfounded. We say $\mathcal{T}$ *drops at* $\alpha + 1$, and put $\alpha + 1 \in D^{\mathcal{T}}$, if $\mathcal{P} \neq \mathcal{M}_\beta^{\mathcal{T}}$ in the above. If $\mathcal{P} = \mathcal{M}_\beta^{\mathcal{T}}$, then we let $i_{\beta,\alpha+1}^{\mathcal{T}} : \mathcal{M}_\beta^{\mathcal{T}} \to \mathcal{M}_\alpha^{\mathcal{T}}$ be the canonical embedding. At limit stages $\lambda < \tau$, Player II must pick a cofinal wellfounded branch of the tree $\mathcal{T}$ built so far.

This is a cofinal-in-λ set $b \subseteq \lambda$ which is a branch of $\mathcal{T}$, such that $D^{\mathcal{T}} \cap b$ is finite, and such that $\mathcal{M}_b^{\mathcal{T}}$ is wellfounded, where $\mathcal{M}_b^{\mathcal{T}}$ is the direct limit of the $\mathcal{M}_\alpha^{\mathcal{T}}$ for $\alpha \in b - \sup(D^{\mathcal{T}} \cap b)$ under the $i_{\alpha\gamma}^{\mathcal{T}}$'s. If II fails to do this, I wins and the game ends. Otherwise, we let $\mathcal{M}_\lambda^{\mathcal{T}} = \mathcal{M}_b^{\mathcal{T}}$, let $i_{\alpha\lambda}^{\mathcal{T}}$ be the canonical embedding for $\alpha \in b - \sup(D^{\mathcal{T}} \cap b)$, and continue. If I does not win at some stage $< \tau$, then II wins.

An *iteration tree* on $\mathcal{M}$ is just a run $\mathcal{T}$ of some $\mathcal{G}(\mathcal{M}, \tau)$ in which II has not lost. A τ*-iteration strategy* for $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}(\mathcal{M}, \tau)$. We say $\mathcal{M}$ is τ*-iterable* just in case there is a τ-iteration strategy for $\mathcal{M}$.

An iteration tree $\mathcal{T}$ is normal iff $lh(E_\beta^{\mathcal{T}}) < lh(E_\gamma^{\mathcal{T}})$ whenever $\beta < \gamma$, and the $\mathcal{T}$-predecessor of $\alpha + 1$ is the least β such that $\text{crit}(E_\alpha^{\mathcal{T}}) < lh(E_\beta^{\mathcal{T}})$, for all α. We say $\mathcal{T}$ is *almost normal* if it is a linear composition of normal trees. We require no more than almost normal iteration trees in this paper.

The importance of being iterable resides in the following lemma and its proof. Let us say that δ is a *cutpoint of* $\mathcal{M}$ iff for all E on the $\mathcal{M}$-sequence, $\text{crit}(E) \leq \delta \Rightarrow lh(E) < \delta$. We write $\mathcal{N} \trianglelefteq \mathcal{M}$ ("$\mathcal{N}$ is an initial segment of $\mathcal{M}$") iff $\mathcal{N} = \mathcal{J}_\delta^{\mathcal{M}}$ for some cutpoint δ of $\mathcal{M}$.

Comparison Lemma. *Let Σ and Γ be $\tau + 1$-iteration strategies for $\mathcal{M}$ and $\mathcal{N}$ respectively, where $\tau = \max(|\mathcal{M}|, |\mathcal{N}|)^+$. Then there are normal iteration trees $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$, played according to Σ and Γ and having last models $\mathcal{P}$ and $\mathcal{Q}$ respectively, such that $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$. Moreover, if $\mathcal{P} \trianglelefteq \mathcal{Q}$ then the branch of $\mathcal{T}$ leading from $\mathcal{M}$ to $\mathcal{P}$ has no drops, and if $\mathcal{Q} \trianglelefteq \mathcal{P}$ then the branch of $\mathcal{U}$ from $\mathcal{N}$ to $\mathcal{Q}$ has no drops.*

§3. The proof of the main result. For simplicity, we shall outline a proof that $\text{HOD}^{L(\mathbb{R})}$ is a core model below $(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$ which uses a mouse-existence hypothesis slightly stronger than $\text{AD}^{L(\mathbb{R})}$. The proof which uses only $\text{AD}^{L(\mathbb{R})}$ is similar.

The idea of our proof is that $\text{HOD}^{L(\mathbb{R})}$ is the direct limit of a certain class $\mathcal{F}$ of countable, iterable mice, under the iteration maps given by the comparison process. The mice in $\mathcal{F}$ are "small" enough that they have $\omega_1 + 1$-iteration strategies in $L(\mathbb{R})$. They are as "full" as possible, granted this smallness condition. (One gets a typical member of $\mathcal{F}$ by iterating M_ω, the minimal iterable class-size premouse with ω Woodin cardinals, then cutting the iterate off at a successor cardinal below its bottom Woodin cardinal.)

Definition 3.1. Let $\mathcal{M}$ be a premouse; then $\mathcal{M}$ is ω*-small* iff whenever $\kappa = \text{crit}(E)$ for some E on the $\mathcal{M}$-sequence, then $\mathcal{J}_\kappa^{\mathcal{M}} \neq$ "There are ω Woodin cardinals". $\mathcal{M}$ is *properly small* iff $\mathcal{M}$ is ω-small, and $\mathcal{M}$ satisfies ZF^-+ "There are no Woodin cardinals" + "There is a largest cardinal".

If $\mathcal{M}$ is ω-small, so is any $\mathcal{J}_\beta^{\mathcal{M}}$; on the other hand, $\mathcal{M}$ may be properly small, and yet for some β, $\mathcal{J}_\beta^{\mathcal{M}} \models$ "There are ω Woodin cardinals".

It is easy to see that if $\mathcal{M}$ is countable and $L(\mathbb{R}) \models \Sigma$ is an ω_1-iteration strategy for $\mathcal{M}$, then Σ really is an ω_1-iteration strategy for $\mathcal{M}$. We have a converse for properly small $\mathcal{M}$:

THEOREM 3.2. *Assume $ZFC + AD^{L(\mathbb{R})}$. Let $\mathcal{M}$ be countable, properly small, and ω_1-iterable (in V). Then*

(a) $L(\mathbb{R}) \models \mathcal{M}$ *is* $\omega_1 + 1$*-iterable,*

(b) *every countable iteration tree $\mathcal{T}$ on $\mathcal{M}$ has at most one cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ is ω_1-iterable.*

We sketch a proof of 3.2 for the reader familiar with [13]. Since $\mathcal{M}$ is ω_1-iterable, II wins the weak iteration game $W\mathcal{G}(\mathcal{M}, \omega)$. This is a $\mathrm{G}^{\mathbb{R}}\Pi^1_1$ assertion about $\mathcal{M}$, hence absolute to $L(\mathbb{R})$. (See [8].) But [13] shows that this weak iterability is enough to compare properly small mice. An argument from [10, §5] then shows that, in $L(\mathbb{R})$, $\mathcal{M}$ is ω_1-iterable via the strategy of choosing the unique cofinal branch b such that II wins $W\mathcal{G}(\mathcal{M}_b^{\mathcal{T}}, \omega)$. This (unique) ω_1-iteration strategy for $\mathcal{M}$ extends to what is, in $L(\mathbb{R})$, an $\omega_1 + 1$ iteration strategy for $\mathcal{M}$. (At stage ω_1, choose the unique cofinal b such that in $L(\mathbb{R})^{\mathrm{Col}(\omega,\omega_1)}$, II wins $W\mathcal{G}(\mathcal{M}_b^{\mathcal{T}}, \omega)$. $\mathrm{AD}^{L(\mathbb{R})}$ implies there is a generic $j : L(\mathbb{R}) \to L(\mathbb{R})^{\mathrm{Col}(\omega,\omega_1)}$, by an unpublished result of Woodin, hence there is such a b in $L(\mathbb{R})^{\mathrm{Col}(\omega,\omega_1)}$. There is such a b in $L(\mathbb{R})$ by the homogeneity of $\mathrm{Col}(\omega, \omega_1)$. This is where we use $\mathrm{AD}^{L(\mathbb{R})}$.)

By 3.2, a countable, properly small, ω_1-iterable $\mathcal{M}$ has exactly one ω_1-iteration strategy. We call countable tree $\mathcal{T}$ on $\mathcal{M}$ *correct* iff $\mathcal{T}$ is played according to this strategy. Equivalently, $\mathcal{T}$ is correct iff every model $\mathcal{M}_\alpha^{\mathcal{T}}$ on $\mathcal{T}$ is ω_1-iterable. By 3.2, correctness is a $\Sigma_1^{L(\mathbb{R})}$ property. We say that $\mathcal{M}$ iterates correctly to $\mathcal{N}$ iff there is an almost normal, correct $\mathcal{T}$ on $\mathcal{M}$ with last model $\mathcal{N}$, and such that the branch $\mathcal{M}$-to-$\mathcal{N}$ of $\mathcal{T}$ has no drops. There may in fact be more than one iteration tree witnessing that $\mathcal{M}$ iterates correctly to $\mathcal{N}$, but by the Dodd-Jensen lemma, any two such trees determine the same iteration map $\pi : \mathcal{M} \to \mathcal{N}$. Because properly small $\mathcal{M}$ satisfy ZF^-, π is fully elementary.

DEFINITION 3.3. A countable, properly small premouse $\mathcal{M}$ is *full* iff whenever $\mathcal{M}$ iterates correctly to $\mathcal{P}$, A is a bounded subset of $\mathrm{OR} \cap \mathcal{P}$, and A is $\mathrm{OD}^{L(\mathbb{R})}(\{\mathcal{P}\})$, then $A \in \mathcal{P}$.

Fullness is a $\Pi_1^{L(\mathbb{R})}$ property. One can show (cf. [13, §4]) that $\mathcal{M}$ is full iff whenever $\mathcal{M}$ iterates correctly to $\mathcal{P}$, and $\mathcal{P} \trianglelefteq \mathcal{Q}$ where $\mathcal{Q}$ is ω-small and ω_1-iterable, then $\rho_\omega(\mathcal{Q}) \geq \mathrm{OR} \cap \mathcal{P}$.

We can now define our direct limit system. Set

$$\mathcal{F} = \{\mathcal{M} \mid \mathcal{M} \text{ is countable, properly small, } \omega_1\text{-iterable, and full}\},$$

and for $\mathcal{M}, \mathcal{N} \in \mathcal{F}$, let $\mathcal{M} \prec^* \mathcal{N}$ iff $\exists \mathcal{P}\, (\mathcal{M}$ iterates correctly to $\mathcal{P} \wedge \mathcal{P} \trianglelefteq \mathcal{N})$. The Dodd-Jensen Lemma implies that if $\mathcal{M} \prec^* \mathcal{N}$, then there is exactly one

$\mathcal{P}$ such that $\mathcal{P} \trianglelefteq \mathcal{N}$ and $\mathcal{M}$ iterates correctly to $\mathcal{P}$, and exactly one iteration map from $\mathcal{M}$ to $\mathcal{P}$. We call this map $\pi_{\mathcal{M},\mathcal{N}}$. Clearly $\mathcal{F}$, $\prec^*$, and the function $(\mathcal{M},\mathcal{N}) \mapsto \pi_{\mathcal{M},\mathcal{N}}$ are $\mathrm{OD}^{L(\mathbb{R})}$.

If δ is a cutpoint of $\mathcal{N}$, then any normal iteration tree on $\mathcal{N}$ decomposes into an iteration tree $\mathcal{T}$ on $\mathcal{J}_\delta^{\mathcal{N}}$ with iteration map σ, followed by an iteration tree $\mathcal{U}$ on the last model of $\mathcal{T}$, with all extenders $E_\alpha^{\mathcal{U}}$ used in $\mathcal{U}$ satisfying $\sigma(\delta) < \mathrm{crit}(E_\alpha^{\mathcal{U}})$. This fact easily implies that $\prec^*$ is transitive, and that if $\mathcal{M} \prec^* \mathcal{N} \prec^* \mathcal{P}$, then $\pi_{\mathcal{M},\mathcal{P}} = \pi_{\mathcal{N},\mathcal{P}} \circ \pi_{\mathcal{M},\mathcal{N}}$.

The comparison lemma and fullness imply that $\prec^*$ is directed. For suppose $\mathcal{M},\mathcal{N} \in \mathcal{F}$, and let $\mathcal{T}$ and U be the correct trees given by the comparison lemma. Let $\mathcal{P}$ and $\mathcal{Q}$ be their respective last models, and suppose e.g., $\mathcal{P} \trianglelefteq \mathcal{Q}$. The comparison lemma implies $\mathcal{M}$-to-$\mathcal{P}$ has no drops, so $\mathcal{M}$ iterates correctly to $\mathcal{P}$. But $\mathcal{M}$ is full, so $\rho_\omega(\mathcal{Q}) \geq \mathrm{OR} \cap \mathcal{P}$. This implies that $\mathcal{N}$-to-$\mathcal{Q}$ has no drops, so $\mathcal{N}$ iterates correctly to $\mathcal{Q}$. Thus $\mathcal{M} \prec^* \mathcal{Q}$ and $\mathcal{N} \prec^* \mathcal{Q}$. A little more work shows $\prec^*$ is countably directed.

Now let $\mathcal{M}_\infty =$ direct limit of $(\mathcal{F}, \prec^*)$ under the $\pi_{\mathcal{M},\mathcal{N}}$'s or rather, let $\mathcal{M}_\infty$ be the transitive collapse of the direct limit, which exists since $\prec^*$ is countably directed. For $\mathcal{M} \in \mathcal{F}$, let $\pi_{\mathcal{M},\infty} : \mathcal{M} \to \mathcal{M}_\infty$ be the canonical embedding. Clearly, $\mathcal{M}_\infty$ is a premouse, and $\mathcal{M}_\infty \in \mathrm{HOD}^{L(\mathbb{R})}$.

A direct computation shows that $\mathrm{OR} \cap \mathcal{M}_\infty \leq (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$. For let $\alpha \in \mathrm{OR} \cap \mathcal{M}_\infty$, and fix $\mathcal{M} \in \mathcal{F}$ so that $\pi_{\mathcal{M},\infty}(\bar{\alpha}) = \alpha$ for some $\bar{\alpha}$. Let $\mathcal{G} = \{\mathcal{P} \mid \mathcal{M}$ iterates correctly to $\mathcal{P}\}$. Then $\mathcal{G} \subseteq \mathcal{F}$, and one can easily check that $\mathcal{G}$ is $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$. Also, the relation R is $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$, where $R(\langle \mathcal{P}, \bar{\beta}\rangle, \langle \mathcal{Q}, \bar{\gamma}\rangle)$ iff ($\mathcal{P}, \mathcal{Q} \in \mathcal{G}$ and $\bar{\beta} \in \mathrm{OR}^{\mathcal{P}}$ and $\bar{\gamma} \in \mathrm{OR}^{\mathcal{Q}}$ and $\pi_{\mathcal{P},\infty}(\bar{\beta}) \leq \pi_{\mathcal{Q},\infty}(\bar{\gamma})$). This is because we can check whether $R(\langle \mathcal{P}, \bar{\beta}\rangle, \langle \mathcal{Q}, \bar{\gamma}\rangle)$ holds by comparing $\mathcal{P}$ with $\mathcal{Q}$. Since every $\beta < \alpha$ is of the form $\pi_{\mathcal{P},\infty}(\bar{\beta})$ for some $\mathcal{P} \in \mathcal{G}$, there is a $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$ prewellorder of HC of order type at least α. Thus $\alpha < (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$.

Now suppose $\mathrm{OR} \cap \mathcal{M}_\infty < (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$. Since $\mathcal{M}_\infty$ can be coded simply by a subset of $\mathrm{OR} \cap \mathcal{M}_\infty$, we then have by the Coding Lemma ([12, Chap. 7]) that for some real x, $\mathcal{M}_\infty$ is coded by a $\Delta_1^{L(\mathbb{R})}(\{x\})$ set of reals. But, essentially via §4 of [13], one can show that the complete $\Sigma_1^{L(\mathbb{R})}$ set of reals is projective in any set of reals coding $\mathcal{M}_\infty$. (Thus the complete $\Sigma_1^{L(\mathbb{R})}$ set of reals is $\Delta_1^{L(\mathbb{R})}$ in a real, a contradiction.) The reason is that $\Sigma_1^{L(\mathbb{R})}$ truth is reducible to truth in symmetric extensions of mice. (This is where our mouse existence assumptions come in.) Call $\mathcal{M}$ a $\langle \varphi, x\rangle$ *witness*, for $\varphi(v)$ a Σ_1 formula and $x \in {}^\omega\omega$, if $\mathcal{M}$ has ω Woodin cardinals with supremum δ and for some symmetric submodel of $\mathcal{M}^{\mathrm{Col}(\omega,<\delta)}$ with set of reals R^*, we have $x \in \mathbb{R}^*$ and $L_\alpha(\mathbb{R}^*) \models \varphi[x]$ for $\alpha = \mathrm{OR} \cap \mathcal{M}$. One can show that if there is a sufficiently iterable $\langle \varphi, x\rangle$ witness, then $L(\mathbb{R}) \models \varphi[x]$. (Cf. [13, §4].) Conversely, if $L(\mathbb{R}) \models \varphi[x]$, then $\exists \mathcal{M} \in \mathcal{F}\, \exists_\beta\, (\mathcal{J}_\beta^{\mathcal{M}}$ is a $\langle \varphi, x\rangle$ witness). This last statement is the mouse existence hypothesis from which our proof proceeds. It follows from, say, the existence of ω Woodin cardinals plus a measurable above them all. (Cf. [13, §4].) It follows that for all φ, x: $L(\mathbb{R}) \models \varphi[x] \Leftrightarrow \exists\beta\exists\mathcal{M}$ ($\mathcal{M}$ is a $\langle\varphi, x\rangle$-witness and $\mathcal{M} \preceq \mathcal{J}_\beta^{\mathcal{M}_\infty}$). This shows the complete $\Sigma_1^{L(\mathbb{R})}$ set of reals is projective in any set of reals coding $\mathcal{M}_\infty$. Thus $\text{OR} \cap \mathcal{M}_\infty = (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$.

One proof that for $\alpha = (\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}$, $V_\alpha \cap \text{HOD}^{L(\mathbb{R})}$ is contained in $\mathcal{M}_\infty$ goes roughly as follows. Let T be the tree on $\omega \times \mathcal{M}_\infty$ of all attempts to build a pair $(x, \pi) \in^\omega \omega \times^\omega \mathcal{M}_\infty$ such that x codes a structure $\mathcal{P}_x$ for the language of premice whose universe is $\omega - \{0\}$, $\pi(0)$ is a successor-cardinal and cutpoint of $\mathcal{M}_\infty$, and $\pi \restriction (\omega - \{0\})$ is an elementary embedding from $\mathcal{P}_x$ into $\mathcal{J}_{\pi(0)}^{\mathcal{M}_\infty}$. One can show that T behaves enough like the tree of a $\Sigma_1^{L(\mathbb{R})}$ scale on a complete $\Sigma_1^{L(\mathbb{R})}$ set that the Becker-Kechris proof goes through, and shows $(V_{\boldsymbol{\delta}_1^2})^{\text{HOD}^{L(\mathbb{R})}} \subseteq L[T]$. But $T \in L[\mathcal{M}_\infty]$, and one can show that $L[\mathcal{M}_\infty] \cap V_{(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}} \subseteq \mathcal{M}_\infty$.

The key properties of T for the Becker-Kechris proof are the following. First, one can compute $\Sigma_1^{L(\mathbb{R})}$ truth using $p[T]$. We have already sketched how to do thiws. Second, for any successor cardinal cutpoint δ of $\mathcal{M}_\infty$, $\exists x \in p[T](\ell_x(0) = \delta)$. To see this, pick any $\mathcal{R} \in \mathcal{F}$ and $\bar{\delta}$ such that $\pi_{\mathcal{R},\infty}(\bar{\delta}) = \delta$. Let $\mathcal{S} = \mathcal{J}_{\bar{\delta}}^{\mathcal{R}}$, and let x be such that $\mathcal{P}_x \cong \mathcal{S}$. Since $\pi_{\mathcal{S},\infty} : \mathcal{S} \to \mathcal{J}_\delta^{\mathcal{M}_\infty}$ is essentially an iteration map, the Dodd-Jensen lemma implies that $\mathcal{S}$ does not embed elementarily into any $\mathcal{J}_\gamma^{\mathcal{M}_\infty}$ with $\gamma < \delta$, so that $\ell_x(0) = \delta$. Third, fix y such that $\mathcal{P}_y \cong \mathcal{R}$ for some $\mathcal{R} \in \mathcal{F}$. Let $I(y) = \{x \in p[T] \mid \ell_x(0) \leq \ell_y(0)\}$; then $I(y)$ is $\Sigma_1^{L(\mathbb{R})}$ in y, uniformly in such y. This is because $x \in I(y)$ iff $\exists\mathcal{S}\exists\beta$ ($\mathcal{R}$ iterates correctly to $\mathcal{S}$ and $\mathcal{P}_x \preceq \mathcal{J}_\beta^{\mathcal{S}}$).

One can also show that $(V_{\boldsymbol{\delta}_1^2})^{\text{HOD}^{L(\mathbb{R})}}$ is contained in $\mathcal{M}_\infty$ without using the Becker-Kechris ideas, but using instead generic elementary embeddings.

Quod erat sketchandum.

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