1. (i) Let $a$ and $b$ be nonzero integers. Then $\gcd(a, b)$ is the greatest integer $d$ such that $d$ divides both $a$ and $b$.

(ii) Let $a$ be odd and $b$ even. One way to show $\gcd(a, b) = \gcd(a + b, a - b)$ is to show that the common divisors of $a$ and $b$ are precisely the common divisors of $a + b$ and $a - b$. This involves two directions.

First, we must show if $d$ divides both $a$ and $b$, then $d$ divides both $a + b$ and $a - b$. This is because, if $d$ divides both $a$ and $b$, then we have $a = n \cdot d$, $b = m \cdot d$, so $a + b = (n + m) \cdot d$ and $a - b = (n - m) \cdot d$. So $d$ divides both $a + b$ and $a - b$.

Next, we must show that if $d$ divides both $a + b$ and $a - b$, then $d$ divides $a$ and $d$ divides $b$. To show $d$ divides $a$, note that $2a = (a + b) + (a - b)$. Since $d$ divides $a + b$ and $d$ divides $a - b$, it follows that $d$ divides $2a$. So we can write $2a = kd$. That means that $k$ is even or $d$ is even (a product of odd numbers cannot be even). However, we can show that $d$ must be odd. Indeed, since $a + b$ is odd (odd plus even equals odd), and $d$ divides $a + b$, it follows that $d$ is odd as well (an even number cannot divide an odd number). Since $d$ is not even, $k$ must be even. So write $k = 2k'$. Substituting this expression into $k$, we get $2a = 2k'd$. Dividing by 2, we get $a = k'd$, so we conclude $d$ divides $a$.

Similarly, we can write $2b = (a + b) - (a - b)$. So $d$ divides $2b$. So since $d$ is odd, $d$ divides $b$ (by an argument similar to above).

In conclusion, we have shown that the common divisors of $a$ and $b$ exactly coincide with the common divisors of $a + b$ and $a - b$. So in particular, the greatest common divisor of $a$ and $b$ equals the greatest common divisor of $a + b$ and $a - b$.

We now show that if $a$ and $b$ are both odd, it’s not true that $\gcd(a, b) \neq \gcd(a + b, a - b)$. Indeed, in this case 2 is a common divisor of $a + b$ and $a - b$ (since both $a + b$ and $a - b$ are even as a sum of odd numbers is even); but 2 is not a common
divisor of \(a\) and \(b\) (since \(a\) and \(b\) are not even). To give a concrete example, let \(a = 3, b = 1\). Then \(gcd(a, b) = 1\) but \(gcd(a + b, a - b) = gcd(4, 2) = 2\).  

2. The fundamental theorem of arithmetic states that any natural number greater than or equal to 2 factors uniquely into a product of primes.

Suppose for contradiction there existed a rational number \(x\) such that \(x^2 = 3\). Since \(x\) is rational, let \(x = \frac{a}{b}\). Assume furthermore that \(gcd(a, b) = 1\) (if \(a\) and \(b\) had a common divisor, we could just cancel it from both numerator and denominator). Now, since \(x^2 = 3\), we have \(\frac{a^2}{b^2} = 3\). So \(a^2 = 3b^2\). So 3 divides \(a^2\). In particular, 3 appears in the prime factorization of \(a^2\). Suppose for contradiction 3 did not divide \(a\). But then we could write \(a = \prod p_i^{r_i}\), where \(p_i\) are primes not equal to 3. But this would imply that \(a^2 = \prod p_i^{2r_i}\). But since \(p_i \neq 3\), this contradicts the fact that 3 appears in the prime factorization of \(a^2\). So in fact 3 divides \(a\). So we may write \(a = 3a'\). Plugging this back into \(a\), we get \((3a')^2 = 3b\), so \(9(a')^2 = 3b^2\), so \(3(a')^2 = b^2\). So 3 divides \(b^2\). By an argument similar to above, we conclude 3 divides \(b\). This contradicts our assumption at the beginning that \(gcd(a, b) = 1\). Therefore a rational solution to \(x^2 = 3\) does not exist.

3. We first note that any number \(k\) is congruent to one of the number \(0, 1, \ldots, n-1\) modulo \(n\). This is by the division algorithm: \(k\) may be written as \(k = q \cdot n + r\), where \(0 \leq r \leq n - 1\), showing that \(k \equiv r \mod n\).

Let \(S = \{a_i : 1 \leq i \leq n + 1\}\). Now, for each \(k\) between 0 and \(n - 1\), let \(S_k\) be the set \(\{a_i : a_i \equiv k \mod n\}\). By the above note, every number is congruent to some number between 0 and \(n - 1\), so \(S = S_0 \cup S_2 \cup \ldots S_{n-1}\). Now, \(S\) has \(n + 1\) elements, whereas there are only \(n\) sets \(S_k\). Therefore one of the sets \(S_k\) must contain at least 2 elements (pigeonhole principle). So fix \(k\) such that \(S_k\) contains two distinct elements \(a_i\) and \(a_j\). Then we have \(a_i \equiv k \equiv a_j \mod n\), so \(a_i \equiv a_j \mod n\) (since congruence is transitive).

Now, for the other part, let \(b_1, \ldots, b_n = 0, \ldots, n - 1\). The claim is that no pair of these numbers is congruent modulo \(n\). Well, suppose \(i\) and \(j\) are two numbers between 0 and \(n - 1\). Without loss of generality, say \(i < j\). Then we have \(j - i \geq 1\). We also have \(j - i \leq n - 1\) since \(j \leq n - 1\) and \(i \geq 0\). But, by definition \(i\) and \(j\) are congruent modulo \(n\) only if \(n\) divides \(j - i\). However, \(1 \leq j - i \leq n - 1\) so \(n\) cannot possibly divide it.

4. The congruence \(15x \equiv 3 \mod 265\) has no solutions. For if there were a solution, then we would have \(15x = 265k + 3\), so \(3 = 15x - 265k\). But this implies \(gcd(15, 265)\) divides 3, which is not true as \(gcd(15, 265) = 5\).
5. Suppose $a$ is an integer coprime to 7. The claim is that for any integer $b$, $gcd(a, b) = gcd(a, 7b)$. As in problem 1, we proceed by showing that the common divisors of $a$ and $b$ are precisely the common divisors of $a$ and $7b$.

First, we must show that if $d$ divides $a$ and $d$ divides $b$, then $d$ divides $a$ and $d$ divides $7b$. This is easy.

Next, we must show that if $d$ divides $a$ and $d$ divides $7b$, then $d$ divides $a$ and $d$ divides $b$. We have that $d$ divides $a$ by assumption, so it remains to show $d$ divides $b$. Since $d$ divides $7b$, we have $7b = kd$. But $d$ divides $a$, so 7 does not divide $d$ (if it did, it would divide $a$, which we're assuming it doesn't). Since $7b = kd$, 7 divides $kd$. But since 7 is prime, this means either 7 divides $k$ or 7 divides $d$. However, we just said 7 does not divide $d$. We conclude that 7 divides $k$. So we can write $k = 7k'$. Substituting this back into the equation $7b = kd$, we get $7b = 7k'd$. Dividing by 7, we get $b = k'd$, so in conclusion $d$ divides $b$, as we wanted to show.

6. Let $p$ be prime and $a, b, r$ be positive integers. Suppose $p^r$ divides $ab$. The claim is that $p^r$ divides $a^r$ or $b^r$. First we note that since $p^r$ divides $ab$, then in particular $p$ divides $ab$. Since $p$ is prime and $p$ divides $ab$, it follows that $p$ divides $a$ or $p$ divides $b$. Suppose without loss of generality that $p$ divides $a$. Then it follows that $a = kp$. So $a^r = k^r p^r$. So $p^r$ divides $a^r$, as desired.

(Sidenote: in fact, we can even show $p^r$ divides $a^2$ or $p^r$ divides $b^2$. This is because we may write $a = p^ka'$, $b = p^lb'$ where $p$ does not divide $a'$ and $p$ does not divide $b'$. Then $ab = p^{k+l}a'b'$ and $p$ does not divide $a'b'$. So, since $p^r$ divides $ab$, we must have $k + l \geq r$. So either $k \geq r/2$ or $l \geq r/2$ (if both $k$ and $l$ were smaller than $r/2$, that would violate $k + l \geq r$). Suppose without loss of generality $k \geq r/2$. Then in fact $a^2 = (p^ka')^2 = p^{2k}(a')^2$ is divisible by $p^r$ since $k \geq r/2$ so $2k \geq r$.)

We note that it’s not necessarily true that $p^r$ divides $a$ or $p^r$ divides $b$. For example, let $a = b = r = p = 2$. Then $p^r = 2^2 = 4$ divides $ab = 2 \cdot 2 = 4$, put $p^r = 4$ divides neither $a$ nor $b$, as both are equal to 2.

7. This is an infinite version of problem 3. As in problem 3, let $S = \{a_i : 1 \leq i\}$ and for each $k$ between 0 and $n-1$, let $S_k$ be the set $\{a_i : a_i \equiv k \pmod{n}\}$. Since every number is congruent to some number between 0 and $n-1$, $S = S_0 \cup S_2 \cup \ldots S_{n-1}$. Now, $S$ has infinitely many elements, whereas there are only finitely many sets $S_k$. Therefore one of the sets $S_k$ must contain infinitely many elements (infinite pigeonhole principle). So fix $k$ such that $S_k$ contains infinitely many elements.
Our subsequence will just be the elements of \( S_k \) listed in order. That is, \( a_{i_m} \) = the \( m \)th element of \( S_k \). These numbers are all congruent to each other since they’re all in \( S_k \), so they are all congruent to \( k \) modulo \( n \).

8. Note that if \( n = m^2 \) is odd, \( m \) must also be odd. Since the problem is modulo 8, it’s enough to consider the values of \( m \) between 0 and 7. Here the only odd numbers are 1, 3, 5 and 7. Squaring them, we get \( 1^2 \equiv 1 \pmod{8}, \ 3^2 \equiv 9 \equiv 1 \pmod{8}, \ 5^2 \equiv 25 \equiv 1 \pmod{8}, \ 7^2 \equiv 49 \equiv 1 \pmod{8} \).

Thus for every odd number \( m \) between 0 and 7, \( m^2 \equiv 1 \pmod{8} \). This gives us the general result: suppose \( n = m^2 \), where \( n \) is odd (so \( m \) is odd). Then let \( m' \) be the remainder upon dividing \( m \) by 8. Since \( m \) is odd, \( m' = m - 8k \) must be odd. Also by the definition of the remainder, \( 0 \leq m' \leq 7 \). But as we checked by hand above, this implies \( (m')^2 \equiv 1 \pmod{8} \). Now since \( m \equiv m' \pmod{8} \), it follows that \( n \equiv m^2 \equiv (m')^2 \equiv 1 \pmod{8} \).

9. We argue by contrapositive. Suppose \( \gcd(a, b) > 1 \). Then there is a number \( d > 1 \) such that \( d \) divides \( a \) and \( d \) divides \( b \). So write \( a = dk, b = dl \). Then \( a^3 = d^3k^3, b^3 = d^3l^3 \). So \( d^3 > 1 \) is a common divisor of \( a^3 \) and \( b^3 \), so \( \gcd(a^3, b^3) > 1 \). Thus, if \( \gcd(a^3, b^3) = 1 \), it follows that \( \gcd(a, b) = 1 \).

10. The number 269 is prime. In general, if we show that all primes less than or equal to \( \sqrt{n} \) do not divide \( n \), we can conclude that \( n \) is prime. This is because if \( n \) were not prime, then since \( n \) would have at least 2 prime factors they cannot all be greater than \( \sqrt{n} \). Now, \( 17^2 = 289 > 269 \), so \( \sqrt{269} < 17 \). So we need only check divisibility by the primes less than 17, i.e. 2, 3, 5, 7, 11, 13. We check and see that 269 is divisible by none of these numbers, so 269 is prime. Since 269 is prime, all numbers between 1 and 268 are relatively prime to 269. In particular, \( \gcd(268, 269) = 1 \). Recall that an equation \( ax \equiv b \pmod{n} \) has a solution iff \( \gcd(a, n) \) divides \( b \), and if there is a solution the number of solutions is \( \gcd(a, n) \). Since \( \gcd(268, 269) = 1 \), the equation \( 268x \equiv 199 \pmod{269} \) has a unique solution.

(Sidenote: if we wanted to find the solution, we would compute the inverse of 268 modulo 269. Since \( 268 \equiv -1 \pmod{269} \), it follows that \( 268^2 \equiv (-1)^2 \equiv 1 \pmod{269} \), so 268 is its own inverse. So we multiply both sides of the equation \( 268x \equiv 199 \pmod{269} \) by \( 268 \equiv -1 \) to get \( x \equiv -1 \cdot 199 \equiv 70 \pmod{269} \)).