Section 5.11, Problem 10: Show that the fourth-order Runge-Kutta method,

\[
\begin{align*}
    k_1 &= hf(t_i, w_i), \\
    k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right), \\
    k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right), \\
    k_4 &= hf(t_i + h, w_i + k_3), \\
    w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{align*}
\]

when applied to the differential equation \(y' = \lambda y\), can be written in the form

\[
    w_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right)w_i.
\]

Solution: We have

\[
\begin{align*}
    w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
             &= w_i + \frac{1}{6}hf(t_i, w_i) + \frac{1}{3}hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf(t_i, w_i)\right) \\
             &\quad + \frac{1}{3}hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf(t_i, w_i)\right) \\
             &\quad + \frac{1}{6}hf\left(t_i + h, w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}hf(t_i, w_i)\right)\right).
\end{align*}
\]

Since \(y' = \lambda y = f(t, y)\), we have

\[
\begin{align*}
    w_{i+1} &= w_i + \frac{1}{6}h\lambda w_i + \frac{1}{3}h\lambda\left(w_i + \frac{1}{2}h\lambda w_i\right) \\
             &\quad + \frac{1}{3}h\lambda\left(w_i + \frac{1}{2}h\lambda\left(w_i + \frac{1}{2}h\lambda w_i\right)\right) \\
             &\quad + \frac{1}{6}h\lambda\left(w_i + h\lambda\left(w_i + \frac{1}{2}h\lambda\left(w_i + \frac{1}{2}h\lambda w_i\right)\right)\right) \\
             &= w_i \left(1 + \frac{1}{6}h\lambda + \frac{1}{3}h\lambda + \frac{1}{2}(h\lambda)^2 \right. \\
             &\quad + \frac{1}{3}h\lambda + \frac{1}{6}(h\lambda)^2 + \frac{1}{12}(h\lambda)^3 \\
             &\quad + \frac{1}{6}h\lambda + \frac{1}{6}(h\lambda)^2 + \frac{1}{12}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4 \right) \\
             &= \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right)w_i. \quad \checkmark
\end{align*}
\]


**Section 5.11, Problem 15(a):** Show that the Implicit Trapezoidal method

\[ w_0 = \alpha, \]
\[ w_{i+1} = w_i + \frac{h}{2} \left[ f(t_{i+1}, w_{i+1}) + f(t_i, w_i) \right], \]

is $A$-stable.

**Solution:** The region $R$ of absolute stability is $R = \{ h\lambda \in \mathbb{C} \mid |Q(h\lambda)| < 1 \}$, where $w_{i+1} = Q(h\lambda)w_i$. A numerical method is said to be $A$-stable if its region of stability $R$ contains the entire left half-plane.

In other words, in order to show that the method is $A$-stable, we need to show that when it is applied to the scalar test equation $y' = \lambda y = f$, whose solutions tend to zero for $\lambda < 0$, all the solutions of the method also tend to zero for a fixed $h > 0$ as $i \to \infty$.

For the Implicit Trapezoidal method, we have

\[ w_{i+1} = w_i + \frac{h}{2}(\lambda w_{i+1} + \lambda w_i), \]
\[ w_{i+1} - \frac{h\lambda}{2}w_{i+1} = w_i + \frac{h\lambda}{2}w_i, \]
\[ w_{i+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}w_i, \]
\[ w_{i+1} = \left( \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right)^{n+1}w_0. \]

Thus,

\[ Q(h\lambda) = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} = \frac{2 + h\lambda}{2 - h\lambda}. \]

Note that for $Re(h\lambda) < 0$, $|Q(h\lambda)| < 1$, and for $Re(h\lambda) > 0$, $|Q(h\lambda)| > 1$. Therefore, the region of absolute stability $R$ for the Implicit Trapezoidal methods is the entire left half-plane, and hence, the method is $A$-stable.

**Section 5.11, Problem 7(b):** Solve the following stiff initial-value problem using the Trapezoidal Algorithm with $TOL = 10^{-5}$

\[ y' = -20(y-t)^2 + 2t, \quad 0 \leq t \leq 1, \]
\[ y(0) = \frac{1}{3}, \]

with $h = 0.1$. Compare the results with the actual solution $y(t) = t^2 + \frac{1}{3}e^{-20t}$.

**Solution:** Slightly modifying the code I posted on my homepage for the problem above and running it gives the following results:

$N = 10$, $h = 0.1$, $t = 1.0$, $w = 1.0488$, $y = 1.0000$, $error = 4.87754e - 002$. 


Section 5.6, Problem 6(a): THERE IS A TYPO IN THE BOOK. THE SOLUTION TO THE INITIAL VALUE PROBLEM DOES NOT MATCH THE ACTUAL SOLUTION. WE WILL BE USING A DIFFERENT ODE.

Use Adams Fourth-Order Predictor-Corrector algorithm of section 5.6 to approximate the solutions to the initial-value problem

\[ \begin{align*}
  y' &= t^2 - 2e^{-2t}, \quad 0 \leq t \leq 1, \\
  y(0) &= 1,
\end{align*} \]  

(3)

with \( h = 0.1 \). Compare the results with the actual solution \( y(t) = \frac{t^3}{3} + e^{-2t} \).

**Solution:** For this problem, we compute starting values \( w_i, i = 1, 2, 3 \) using the fourth order Runge-Kutta method:

\[ \begin{align*}
  k_1 &= hf(t_i, w_i), \\
  k_2 &= hf(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}), \\
  k_3 &= hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}), \\
  k_4 &= hf(t_i + h, w_i + k_3), \\
  w_{i+1} &= w_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\] \]  

(4)

For \( i = 4, 5, \ldots \) we use Adams Fourth-Order Predictor-Corrector method, which consists of the predictor Adams-Bashforth, and corrector Adams-Moulton techniques.

The fourth-order Adams-Bashforth technique, an explicit four-step method, is defined as:

\[ \begin{align*}
  w_{i+1} &= w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})].
\] \]  

(5)

The fourth-order Adams-Moulton technique, an implicit three-step method, is defined as:

\[ \begin{align*}
  w_{i+1} &= w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})].
\] \]  

(6)

Running the Adams Fourth-Order Predictor-Corrector algorithm gives the following results at the final step:

\[ \begin{align*}
  N = 10, h = 1.0000000e - 001, t = 1.00, \\
  w = 4.6864787414e - 001, y = 4.68666861657e - 001, error = 2.0742429498e - 005.
\] \]

You can verify that the solutions obtained with the method are indeed satisfying the fourth order accuracy. Check this, for example, running the code with \( h = 0.01 \) and \( h = 0.005 \) and calculate the order of convergence using the formula from homework 2.
Section 5.6, Problem 12: Derive the Adams-Bashforth three-step explicit method

\[ w_{i+1} = w_i + \frac{h}{12} [23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})] \] (7)

by the following method. Set

\[ y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2})). \] (8)

Expand \( y(t_{i+1}) \), \( f(t_{i-2}, y(t_{i-2})) \), and \( f(t_{i-1}, y(t_{i-1})) \) in Taylor series about \((t_i, y(t_i))\), and equate the coefficients of \( h, h^2, \) and \( h^3 \) to obtain \( a, b, \) and \( c \).

Solution: Since \( y'(t_i) = f(t_i, y(t_i)) \), we can write equation (8) as

\[ y(t_{i+1}) = y(t_i) + ah'y(t_i) + bh'y(t_{i-1}) + ch'y(t_{i-2}). \] (9)

Expanding both sides of (9) in Taylor series about \( t_i \), we obtain

\[
y(t_i) + hy'(t_i) + \frac{1}{2}h^2y''(t_i) + \frac{1}{6}h^3y'''(t_i) + O(h^4) \\
= y(t_i) + ah'y(t_i) + bh\left(y'(t_i) - hy''(t_i) + \frac{1}{2}h^2y'''(t_i) + O(h^3)\right) \\
+ ch\left(y'(t_i) - 2hy''(t_i) + \frac{3}{2}h^2y'''(t_i) + O(h^3)\right), \\
= y(t_i) + (a + b + c)hy'(t_i) + (-b - 2c)h^2y''(t_i) + \left(\frac{1}{2}b + 2c\right)h^3y'''(t_i) + O(h^4).
\]

Thus, equating the coefficients, we obtain

\[
1 = a + b + c, \\
\frac{1}{2} = -b - 2c, \\
\frac{1}{6} = \frac{1}{2}b + 2c,
\]

which gives \( a = \frac{23}{12}, b = -\frac{16}{12}, c = \frac{5}{12} \). Plugging these into (8), we obtain

\[ y(t_{i+1}) = y(t_i) + \frac{h}{12} [23f(t_i, y(t_i)) - 16f(t_{i-1}, y(t_{i-1})) + 5f(t_{i-2}, y(t_{i-2}))] + O(h^4), \]
or

\[ w_{i+1} = w_i + \frac{h}{12} [23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})]. \]

The order of the local truncation for the Adams-Bashforth three-step explicit method is, therefore, \( \tau(h) = O(h^3) \).