“Coordinate Update Algorithm Short Course”

Operator Splitting on Diagrams

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Forward operator

- $A: \mathcal{H} \rightarrow \mathcal{H}$ is single valued
- forward operator:
  \[ F_{\gamma A} := I - \gamma A \]
- diagram:
  
  \[
  z \quad \overset{-\gamma A(z)}{\longmapsto} \quad F_{\gamma A}(z)
  \]
Backward operator

- $A : \mathcal{H} \Rightarrow \mathcal{H}$, possibly multi-valued
- **backward (resolvent) operator:**
  \[
  J_{\gamma A} := (I + \gamma A)^{-1}
  \]
- **implicit step:**
  \[
  z' := J_{\gamma A} z \iff z \in (I + \gamma A)z' \\
  \iff \exists \tilde{A}(z') \in A(z') \text{ s.t. } z = z' + \gamma \tilde{A}(z')
  \]
- **diagram:**
  \[
  z \rightarrow \rightarrow \rightarrow z' = J_{\gamma A}(z)
  \]
Reflection operator (Reflected backward operator)

- **reflection operator:**
  \[ R_{\gamma A} := I + 2(J_{\gamma A} - I) = 2J_{\gamma A} - I \]

- Let \( z' := J_{\gamma A}(z) \)

- **diagram:**

\[ \begin{array}{ccc}
  z & \longrightarrow & z' = J_{\gamma A}(z) \\
  -\gamma \tilde{A}(z') & \longrightarrow & -\gamma \tilde{A}(z') \\
  R_{\gamma A}(z) & \longrightarrow & R_{\gamma A}(z)
\end{array} \]
Line styles

- forward
- resolvent
- reflection ext.
- assistant
Forward-backward splitting

- $A$ is maximally monotone
- $B$ is cocoercive
- **forward-backward splitting (FBS) algorithm:**

$$
x^{k+1} = J_{\gamma A} \circ F_{\gamma B}(x^k)
$$

**fixed-point encodes zero:**

$$
x^k = x^{k+1} \implies 0 = Bx^k + \tilde{A}x^{k+1} \implies 0 \in Bx^k + Ax^k
$$
remarks:

- ✓ $x^* = J_{\gamma A} \circ F_{\gamma B}(x^*) \iff 0 \in Ax^* + Bx^*$

- ✓ $J_{\gamma A} \circ F_{\gamma B}$ is averaged, since
  - cocoercive $B +$ proper $\gamma \Rightarrow F_{\gamma B}$ is averaged
  - maximally monotone $A \Rightarrow J_{\gamma A}$ is averaged
  - averagedness is closed under composition

- ✓ what if $B$ is not cocoercive?
Double-backward splitting

- $A, B$ are maximally monotone
- double-backward splitting (DBS) algorithm:

$$x^{k+1} = J_{\gamma A} \circ J_{\gamma B}(x^k)$$

fixed-point does not encode zero:

$$x^k = x^{k+1} \implies 0 = B x^{k+1/2} + A x^{k+1} \implies 0 \in B x^{k+1/2} + A x^k$$
remarks:

- ✓ $J_{\gamma A} \circ J_{\gamma B}$ is averaged, since
  - maximally monotone $A, B \Rightarrow J_{\gamma A}, J_{\gamma B}$ are both averaged
  - averagedness is closed under composition

- ✗ $x^* = J_{\gamma A} \circ J_{\gamma B}(x^*) \iff 0 \in Ax^* + Bx^*$
  - exception: $A$ and $B$ share fixed-points, e.g., find a point in $C_1 \cap C_2$
Peaceman-Rachford and Douglas-Rachford splittings

- $A, B$ are maximally monotone
- Peaceman-Rachford splitting (PRS) uses $\alpha = 1$:
  
  \[
  z^{k+1} = (1 - \alpha)z^k + \alpha R_{\gamma A} \circ R_{\gamma B}(z^k)
  \]

  Douglas-Rachford splitting (DRS) uses $\alpha = 1/2$:

  \[
  z^{k+1} = \frac{1}{2}z^k + \gamma \tilde{B}x_B^k + (1 - \gamma)\tilde{A}x_A^k
  \]

  fixed-point encodes zero:

  \[
  z^k = z^{k+1} \implies 0 = \tilde{B}x_B^k + \tilde{A}x_A^k \implies x_B^k = x_A^k \implies 0 \in Bx_B^k + Ax_B^k
  \]
remarks:

- $z^* = R_{\gamma A} \circ R_{\gamma B}(z^*) \iff 0 \in Ax^* + Bx^*$
  - through $x^* = J_{\gamma B}(z^*)$

- $R_{\gamma A} \circ R_{\gamma B}$ is nonexpansive, since
  - $A, B$ are both maximally monotone
    - $\Rightarrow J_{\gamma A}, J_{\gamma B}$ are both $\frac{1}{2}$-averaged
    - $\Rightarrow R_{\gamma A}, R_{\gamma B}$ are both nonexpansive
  - nonexpansiveness is closed under composition

- $\frac{1}{2} I + \frac{1}{2} R_{\gamma A} \circ R_{\gamma B}$ is averaged
Davis-Yin three-operator splitting’15

- $A, B$ are maximally monotone
- $C$ is cocoercive
- **problem:** $0 \in A(x) + B(x) + C(x)$
- **iteration:** $z^{k+1} = (I - J_{\gamma B})z^k + J_{\gamma A}(R_{\gamma B} - \gamma C \circ J_{\gamma B})z^k$

**fixed-point encodes zero:**

$z^k = z^{k+1} \Rightarrow 0 = (\tilde{B} + C)x_B^k + Ax_A^k \Rightarrow x_B^k = x_A^k \Rightarrow 0 \in (A + B + C)x_B^k$
remarks:

- ✓ \( z^* = T_{DYS}(z^*) \iff 0 \in Ax^* + Bx^* + Cx^* \)
  - through \( x^* = J_{\gamma B}(z^*) \)

- ✓ \( T_{DYS} \) is averaged
  - due to “+” and “−”, composition closedness is not enough for proof
  - the proof is short but not straightforward

- special cases: forward-backward, backward-forward, Douglas-Rachford splitting schemes

- benefit: avoid product-space trick or primal-dual splitting, or reduce them
Open questions

- $A, B, C$ are maximally monotone

- **problem:** $0 \in A(x) + B(x) + C(x)$

- **question:** find $T$ such that
  - $z^* = Tz^*$ recovers $x^*$
  - $T$ is averaged
  - $J_{\gamma A}, J_{\gamma B}, J_{\gamma C}$ is applied once each
  - no dummy variables (or very few)

- what about more than three ... ?