“Coordinate Update Algorithm Short Course”
Operator Splitting by Algebra

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Forward operator

- $A : \mathcal{H} \rightarrow \mathcal{H}$ is single valued

- Forward operator:
  \[ F_{\gamma A} := I - \gamma A \]

- For simplicity, set $\gamma = 1$.

- Fixed-point encoding:
  \[ 0 = Ax \iff x = x - Ax \]
  \[ \iff x = F_A(x) \]
Backward operator

- $A : \mathcal{H} \rightrightarrows \mathcal{H}$, possibly multi-valued
- Backward (resolvent) operator:

$$J_{\gamma A} := (I + \gamma A)^{-1}$$

- $J_{\gamma A}(x)$ is an implicit step:

$$z' = J_{\gamma A}z \iff z' = (I + \gamma A)^{-1}z$$

$$\iff z \in (I + \gamma A)z'$$

$$\iff \exists \tilde{A} \in Az', \exists \ z = z' + \gamma \tilde{A}$$
• For simplicity, consider $\gamma = 1$

• Fixed-point encoding:

\[
0 \in Ax \iff x \in x + Ax
\]
\[
\iff x \in (I + A)x
\]
\[
\iff (I + A)^{-1}(x) = x
\]
\[
\iff J_A(x) = x
\]
Reflection operator (Reflected backward operator)

- **reflection operator:**

  \[ R_{\gamma A} := I + 2(J_{\gamma A} - I) = 2J_{\gamma A} - I \]

- **Fixed-point encoding (for \( \gamma = 1 \):**

  \[ 0 \in Ax \iff J_{\gamma A}(x) = x \]

  \[ \iff 2J_{\gamma A}(x) = 2x \]

  \[ \iff 2J_{\gamma A}(x) - x = x \]

  \[ \iff R_{\gamma A}(x) = x \]
Forward-backward splitting (FBS)

- $A$ is maximally monotone, possibly set-valued
- $B$ is single-valued
- Problem:

\[ 0 \in Ax + Bx \]

- Forward-backward splitting (FBS) operator:

\[ T_{FBS} = J_{\gamma A} \circ F_{\gamma B} \]

- Fixed-point encoding (for $\gamma = 1$):

\[ 0 \in Ax + Bx \iff -Bx \in Ax \]
\[ \iff x - Bx \in x + Ax \]
\[ \iff F_A(x) \in (I + A)x \]
\[ \iff J_A(F_A(x)) = x \]
\[ \iff T_{FBS}(x) = x \]
Backward-forward splitting (BFS)

- $A$ is maximally monotone, possibly set-valued
- $B$ is single-valued

Problem:

\[ 0 \in Ax + Bx \]

Backward-forward splitting (BFS) operator:

\[ T_{BFS} = F_{\gamma B} \circ J_{\gamma A} \]

For any $p \in Ax$, define

\[ z := x + p \in (I + A)x \]

Then, $z$ uniquely recover

\[ x = J_A(z) \]
\[ p = z - x = z - J_A(z) \]

Hence, $z$ encodes both $x$ and $p$ even though $Ax$ is multi-valued.
Fixed-point encoding (for $\gamma = 1$):

$$0 \in Ax + Bx \Leftrightarrow -Bx \in Ax$$

$$(z := x - Bx, \; x = J_A(z))$$

$$\Leftrightarrow -Bx = z - J_A(z)$$

$$\Leftrightarrow -B(J_A(z)) = z - J_A(z)$$

$$\Leftrightarrow J_A(z) - B(J_A(z)) = z$$

$$\Leftrightarrow T_{BFS}(z) = z$$

Recover $x = J_A(z)$
Peaceman-Rachford and Douglas-Rachford splittings

- $A, B$ are maximally monotone, possibly set-valued
- Problem:
  \[ 0 \in Ax + Bx \]
- Peaceman-Rachford splitting (PRS) operator:
  \[ T_{PRS} := R_{\gamma A} \circ R_{\gamma B} \]
- Douglas-Rachford splitting (DRS) operator:
  \[ T_{DRS} := \frac{1}{2} I + \frac{1}{2} R_{\gamma A} \circ R_{\gamma B} \]
• Recall reflection: \( R = 2J - I \)

• Fixed-point encoding (for \( \gamma = 1 \)):

\[
0 \in Ax + Bx \iff \exists p, \exists -p \in Ax, p \in Bx
\]

\[
(z := x - p, \ x = J_A(z))
\]

\[
\iff -p = z - J_A(z), \ p \in Bx
\]

\[
\iff 2J_A(z) - z \in J_A(z) + B(J_A(z))
\]

\[
\iff R_A(z) \in (I + B)(J_A(z))
\]

\[
\iff J_B(R_A(z)) = J_A(z)
\]

\[
\iff 2J_B(R_A(z)) = 2J_A(z)
\]

\[
\iff 2J_B(R_A(z)) - R_A(z) = 2J_A(z) - R_A(z)
\]

\[
\iff R_A(R_A(z)) = z
\]

Recover \( x = J_A(z) \)
Davis-Yin three-operator splitting

- $A, B$ are maximally monotone
- $C$ is single-valued

Problem:

$$0 \in Ax + Bx +Cx$$

Davis-Yin splitting operator:

$$T_{DYS} = (I - J_{\gamma B}) + J_{\gamma A} \circ (R_{\gamma B} - \gamma C \circ J_{\gamma B})$$

**homework:** Fixed-point encoding (for $\gamma = 1$)