HOMEWORK 1

Due on Monday, March 1st, in class.

Exercise 1. Let $d = 3$ and $1 < p < 2$. Prove that the initial value problem
\[
\begin{align*}
t_{tt} - \Delta u \pm |u|^p u &= 0 \\
u(0) &= u_0, \quad u_t(0) = u_1
\end{align*}
\]
is locally wellposed for initial data $(u_0, u_1) \in H^1_x \times L^2_x$.

Exercise 2. (Standard blowup criterion) Let $d = 3$ and $2 \leq p \leq 4$. Let initial data $(u_0, u_1)$ belong to the critical homogeneous space $H^s_x \times H^{s-1}_x$. (Recall $s_c = \frac{3}{2} - \frac{2}{p}$.)

Let $u : [0, T_0) \times \mathbb{R}^3 \to \mathbb{R}$ be the unique strong solution to
\[
\begin{align*}
t_{tt} - \Delta u \pm |u|^p u &= 0 \\
u(0) &= u_0, \quad u_t(0) = u_1
\end{align*}
\]
Assume that
\[
\|u\|_{L^{2p}_{t,x}([0, T_0) \times \mathbb{R}^3)} < \infty.
\]
Prove that there exists $\delta > 0$ such that $u$ extends to a strong solution on $[0, T_0 + \delta]$.

Exercise 3. (Scattering, part 1) Let $d = 3$ and $2 \leq p \leq 4$. Let $(u_0, u_1) \in H^s_x \times H^{s-1}_x$. Prove that there exists a unique solution $u : [T, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ to
\[
\begin{align*}
t_{tt} - \Delta u \pm |u|^p u &= 0 \\
u(t) &= u_0, \quad u_t(t) = u_1
\end{align*}
\]
for some $T \in \mathbb{R}$ such that $(u(t), u_x(t)) \in C_t([T, \infty); H^s_x \times H^{s-1}_x)$, $u \in L^{2p}_{t,x}([T, \infty) \times \mathbb{R}^3)$ and
\[
\|u(t) - \cos(t|\nabla|)u_0 - \frac{\sin(t|\nabla|)}{|\nabla|}u_1\|_{H^s_x} \to 0 \quad \text{as} \quad t \to \infty.
\]

Exercise 4. (Scattering, part 2) Let $d = 3$ and $2 \leq p \leq 4$. Let $(u_0, u_1) \in H^s_x \times H^{s-1}_x$. Prove that if
\[
\|(u_0, u_1)\|_{H^s_x \times H^{s-1}_x} \leq \eta
\]
for $\eta = \eta(p)$ sufficiently small, then there exists a unique global solution $u$ to
\[
\begin{align*}
t_{tt} - \Delta u \pm |u|^p u &= 0 \\
u(0) &= u_0, \quad u_t(0) = u_1
\end{align*}
\]
satisfying $(u, u_x) \in C_t(H^s_x \times H^{s-1}_x)$ and $u \in L^{2p}_{t,x}(\mathbb{R} \times \mathbb{R}^3)$. Moreover, for such small initial data there exist $(u_0^\pm, u_1^\pm) \in H^s_x \times H^{s-1}_x$ so that
\[
\|u(t) - \cos(t|\nabla|)u_0^\pm - \frac{\sin(t|\nabla|)}{|\nabla|}u_1^\pm\|_{H^s_x} \to 0 \quad \text{as} \quad t \to \pm \infty.
\]

Exercise 5. (Domain of dependence for classical solutions) Let $I$ be a time interval containing 0 and let $u, v \in C^2_{t,x}(I \times \mathbb{R}^3)$ be two solutions to
\[
\begin{align*}
t_{tt} - \Delta w \pm |w|^p w &= 0, \quad p > 0
\end{align*}
\]
with initial data $(u_0, u_1)$ and $(v_0, v_1)$, respectively. Prove that if the initial data agree on a ball $\{x \in \mathbb{R}^3 : |x - x_0| < R\}$, then the solutions agree on $\{(t, x) \in I \times \mathbb{R}^3 : |x - x_0| < R - |t|\}$.