second printing: typos and minor changes

p.4, l.-10: Change “f(∂∆)” to “∂f(∆)”
p.4, l.-2: Change “are” to “is”
p.7, l.3: Change “wn” to “w0”

pp.9-10: Change the first sentence in Section 3 to read: “It was P. Montel (1911) who first formulated the notion of a normal family of meromorphic functions and proved the criterion that bears his name.” Montel’s original proof was based on Schottky’s theorem, not on Picard’s modular function.

p.13, l.-4: Change “the the” to “the”
p.14, l.6: Change “2.1” to “3.1”
p.16, Fig.3: Add horizontal bars to the fractions \(\frac{\pi}{2}\) and \(\frac{\arg \mu(z)}{2}\)
p.19, l.-7: Insert “has nonvanishing Jacobian” after “If \(f \in QC^1(k, R)\)”
p.20, l.-9: Change “||U_\mu||” to “||U_\mu||_p” (insert subscript \(p\))
p.20, l.-7: Change “||(I - U_\mu)^{-1}||” to “||(I - U_\mu)^{-1}||_p” (insert subscript \(p\))
p.22, l.-5: Change “the smoothness of \(f\)” to “\(f \in QC^1(k, R)\)”
p.22, l.-2: Change “If \(f\) were \(C^1\) we would” to “If \(f\) were \(C^1\) and \(f_z \neq 0\), we would”
p.30, l.-8: Change “due in this form” to “due essentially in this form”
p.33, l.-13: Change “For \(|z|\) small there is” to “Choose”
p.33, l.-12: Change “\(C|z|^p\)” to “\(C|z|^p\) for \(|z| \leq 1/C\). Then \(|f(z)| \leq |z|\) for \(|z| \leq 1/C\)”
p.33, l.-10: Change “\(\delta\)” to “\(1/C\)”
p.33, lines 1 to 2: Delete “at the origin”
p.34, l.-7: Delete “\(c < \)” so that it reads “\(|z| \leq 1/C\)”
p.36, l.-1: Change comma to period after the last estimate, and add the line: “where the estimate is uniform for \(z\) belonging to a compact set.”
p.39, l.9: Change “to to” to “to”
p.41, l.-14: Change “let” to “suppose \(z_0 = 0\) is a fixed point of \(f(z)\), with multiplier”
p.71, l.-13: Change “inside” to “on the bounded components of”

p.71, l.-3: Change “2d − 1” to “2d + 1”

p.71, l.-2: Change “2d − 1” to “2d + 1”

p.74, l.-1: Change “an isometry” to “a local isometry”

p.75, l.3: Change “an isometry” to “a local isometry”

p.75, l.4: Change “z, w ∈ U. In particular for any” to “z, w ∈ U, z ≠ w. Further, for any”

p.75, l.18: Change “an isometry” to “a local isometry”

p.75, l.-5: Change “By an isometry, we mean at the local level, so that the lift” to “Since f is a local isometry, the lift”

p.76, l.10: After the first sentence, insert “We claim that either (1) or (2) holds. For this, suppose that (2) fails.”

p.77, l.15: Change “(2)” to “(1)”

p.87, l.2: Change “λn−1” to “λn”

p.91, l.14: Change “had been” to “is”

p.91, l.15-16: Delete the sentence “Recently . . . z16 + c.”

p.100, l.3: Change “through” to “around”

p.101, l.10: Change “conjugate to” to “conjugate on U1 to”

p.128, l.15: Change “are dense in M” to “are dense in ∂M”

p.149, format: Insert space at end of example, between lines -5 and -6

p.154, l.-2: Insert “= f(z, c)” after “P_{c}^{\ell}(z)”

p.157, format: Insert space at end of statement of theorem, between lines 16 and 17

p.173, Index: Change “repulsive cycle, ??” to “repulsive cycle, 172”
The very last assertion of the proof requires justification. To do this, we follow the proof given, except that we take ψ to be the universal covering map of the upper half-plane \( \mathbb{H} \) over \( \mathbb{C} \setminus \{0, 1\} \) constructed in the proof of Theorem 3.1, with fundamental domain \( E \) from that proof, and we choose the lifts \( \tilde{f} \) of functions \( f \in F \) so that \( \tilde{f}(0) \in E \). The functions \( \tilde{f} \) still form a normal family. Let \( \{f_n\} \) be a sequence in \( F \). Passing to a subsequence, we can assume that \( \tilde{f}_n \) converges normally to \( g \) on \( \mathbb{H} \). We must show that \( f_n \) converges normally.

If \( \tilde{f}_n(0) \) converges to a point of \( \mathbb{H} \), then the image of \( g \) is in \( \mathbb{H} \), and we can use the local inverses of \( \psi \) to see that \( f_n \) converges normally to the analytic function \( \psi \circ g \). Our problem is to determine what happens when the limit of \( \tilde{f}_n(0) \) does not belong to \( \mathbb{H} \).

If the limit of \( \tilde{f}_n(0) \) is not in \( \mathbb{H} \), then since \( \tilde{f}_n(0) \in E \), either \( \text{Re} \tilde{f}_n(0) \to +\infty \), or \( \tilde{f}_n(0) \) converges to one of the corners 0 or 1 of \( \partial E \). By composing the functions in the family \( F \) with a fractional linear transformation that permutes the points \( 0, 1, \infty \), we can assume that \( \text{Re} \tilde{f}_n(0) \to +\infty \). Then by Harnack’s theorem, \( \text{Re} \tilde{f}_n(z) \to +\infty \) uniformly on compacta.

Using the periodicity of \( \psi \), we see that \( |\psi(w)| \to \infty \) uniformly as \( \text{Re} w \to +\infty \). Hence \( f_n = \psi \circ \tilde{f}_n \) converges to \( \infty \) uniformly on compacta, and in particular it converges normally, as required.

A variant of the proof, which avoids Harnack’s theorem, proceeds in outline as follows. Replacing the family of functions \( F \) by the family of their square roots, one assumes that the family omits four points \( \{-1, 0, 1, \infty\} \) in the extended plane. Then one proceeds as above, to the case where \( \tilde{f}_n(0) \) converges to a vertex of \( E \). In this case one considers the compositions \( g_n = \varphi \circ f_n \), where \( \varphi \) is the fractional linear transformation that maps \( -1 \) to that vertex and leaves the other two vertices of \( E \) fixed. Now \( \tilde{g}_n(0) = \varphi(\tilde{f}_n(0)) \) converges to a point of \( \mathbb{H} \). We conclude as before that \( g_n \) converges normally, as does \( f_n \).

Theorem 1.1 requires some justification to the effect that a neutral fixed point in the Fatou set belongs to a Siegel disk as defined. The following lemma clarifies the definition of a Siegel disk, and Theorem 1.1 follows immediately.

**Lemma.** Let 0 be a neutral fixed point for a rational function \( R \), with multiplier \( \lambda \). If \( 0 \in F \), and if \( U \) is the component of the Fatou set containing 0, then Schröder’s equation \( \varphi(R(z)) = \lambda \varphi(z) \), with side conditions \( \varphi(0) = 0, \varphi'(0) = 1 \), has a (unique) solution \( \varphi(z) \) defined on \( U \) and mapping \( U \) conformally onto a disk.

**Proof.** We can assume that \( \infty \in F \). Note that \( R(U) \subseteq U \). Since the iterates \( R^n \) form a normal family on \( U \), they are uniformly bounded on compact subsets of \( U \). As in the proof of Theorem II.6.2, the functions \( \varphi_n(z) = (1/n) \sum_{j=0}^{n-1} \lambda^{-j} R^j(z) \) are uniformly bounded on compact subsets of \( U \), and any limit \( \varphi(z) \) of the \( \varphi_n \)'s has the required properties. \( \Box \)
To see that one of the alternatives (1), (4), or (5) holds, proceed as follows.

Suppose that $U$ is a punctured disk, say $U = \Delta \setminus \{0\}$, with covering map $\psi(\zeta) = e^{2\pi i \zeta}$ from the upper half-plane $\mathbb{H}$ to $U$. If $f$ is a local hyperbolic isometry of $U$, and $F$ is the lift of $f$ to $\mathbb{H}$, then $F(\zeta + 1) \equiv F(\zeta)$, so there is an integer $m$ such that $F(\zeta + 1) = F(\zeta) + m$. Thus $F$ fixes $\infty$, and $F$ is affine. Evidently $F(\zeta) = m\zeta + b$ where $m \geq 1$ and $b$ is real. Thus $f(z) = e^{2\pi ib}z^m$. If $m > 1$, then (1) holds. If $m = 1$, then since $\Gamma$ is discrete, $b$ is irrational, and (5) holds.

A similar argument shows that if $U$ is an annulus, then (4) holds.

In this proof, the $a$ and $c$ do not come directly from the statement of Lemma 2.1. They come from an open neighborhood $V$ of $J$, as follows. Let $V$ be an $\varepsilon$-neighborhood of $J$ with respect to the hyperbolic metric of $D = \mathbb{C} \setminus CL$. Then $R^{-1}(V) \subset V$. For $\varepsilon > 0$ small, there is $A > 1$ such that (2.1) holds for $z \in V$. Set $c = 1/A$ and $a = (\sup \sigma)/(\inf \sigma)$, where the sup and the inf are taken over $V$. Then $|(R^k)'(z)| \geq a/c^k$ for all $z \in V$ such that $R^k(z) \in V$, as in the proof of Lemma 2.1.

There is a gap in the proof, which requires substantial work to fill. The problem is to show that if $P_a$ has a parabolic cycle, then $\theta$ has odd denominator. The gap is filled, and in a more general setting, in the Doctoral Dissertation of Gustav Ryd, “Iterations of one parameter families of complex polynomials,” Department of Mathematics, KTH, Stockholm (1997), ISBN 91-7170-210-5. The relevant statement is Proposition 5.8, whose proof covers pages 38-43.

Ryd’s thesis contains much more. In particular, it contains (Dissertation Section 3) theorems on the landing of external rays at parabolic and repelling periodic points of the Julia set of a rational function. It also carries out (Dissertation Section 7 and Theorem 8.1) the “main deformation construction” sketched at the end of Section VIII.7, again in a more general setting.

Ryd devotes special attention to one-parameter families of polynomials that have the form

$$P_c(z) = z^d + \alpha_{d-1}(c)z^{d-1} + \ldots + \alpha_0(c), \quad P_c'(z) = d \prod_{j=1}^{d-1} (z - p_j(c)),$$

where $\alpha_0(c), \ldots, \alpha_{d-1}(c)$ and $p_1(c), \ldots, p_{d-1}(c)$ are polynomials in $c$. This includes such one-parameter families such as $z^d + c$, and more generally $p(z) + c$, where $p$ is a polynomial. Thus each critical point has polynomial dependence on $c$, and one can define a “Mandelbrot set” $\mathcal{M}_j$ for each critical point. Ryd investigates the behavior of $P_c$ as $c \to a \in \mathcal{M}_j$. 