Solution to Exercises 6.3.3 through 6.3.5.

6.3.3. (a) The joint density of $Y_1, \ldots, Y_{k-1}$ under $H_i$ for $i \neq 0$ is given by (6.22). Let $Z_1 = |Y_1|, Z_2 = Y_2 \text{sgn} Y_1, \ldots, Z_{k-1} = Y_{k-1} \text{sgn} Y_1$. This is a two-to-one map and the density of $Z_1, \ldots, Z_{k-1}$ is given for $z_i > 0$ by

$$f_{Z_1, \ldots, Z_{k-1}}(z_1, \ldots, z_{k-1}) = f_{Y_1, \ldots, Y_{k-1}}(z_1, \ldots, z_{k-1}) + f_{Y_1, \ldots, Y_{k-1}}(-z_1, \ldots, -z_{k-1})$$

$$= \frac{1}{\sqrt{k(2\pi)^{(k-1)/2}}} \exp\left(-\frac{k^2}{2} \sum_i z_i^2 - \frac{\Delta^2(k-1)}{2k} \right) \left[ \exp\{\Delta(z_i - \bar{z})\} + \exp\{-\Delta(z_i - \bar{z})\} \right]$$

where $s_i^2 = (1/k) \sum_k (z_j - \bar{z})$ and $z_k = 0$.

(b) The term in square brackets is just $2 \cosh(\Delta(z_i - \bar{z}))$ and since $\cosh(x)$ is symmetric in $x$, the density of the $Z$'s depends only on $|\Delta| = \Delta_0 > 0$. Since it is increasing on $[0, \infty)$, we have

$$\phi(0|x) = 1 \quad \text{if max}_j |z_j - \bar{z}| \geq c$$

and $\phi(0|x) = 0$ otherwise. But since $|z_i - \bar{z}| = |x_i - \bar{x}|$ for all $i$, we have

$$\phi(0|x) = \begin{cases} 1 & \text{if max}_j |x_j - \bar{x}| \leq c \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi(i|x) = \begin{cases} 1 & \text{if } |x_i - \bar{x}| = \text{max}_j |x_j - \bar{x}| > c \\ 0 & \text{otherwise} \end{cases}$$

where $c$ is chosen so that $P(\text{accept } H_0|H_0) = 1 - \alpha$. This test is best out of the invariant rules satisfying (a) and (b) of Theorem 1 and does not depend on $\Delta_0$.

6.3.4. (a) The joint distribution of $X_1, \ldots, X_k$ under $H_i$ has density

$$f_{X_1, \ldots, X_k}(x_1, \ldots, x_k|i) = \frac{1}{\Gamma(\alpha)^k \beta^k} \exp\left(-\frac{1}{\beta} \sum x_j\right) \left( \prod x_j \right)^{\alpha-1} \frac{1}{\lambda} \exp\left\{ \frac{x_k(\lambda - 1)}{\beta \lambda} \right\}$$

where all $x_j > 0$. We make the change of variable $Z_j = X_j/X_k$ for $j = 1, \ldots, k-1$ and $Y = X_k$, and we let $Z_k$ be a dummy variable equal to 1. The inverse transformation is $X_j = Y Z_j$ for $j = 1, \ldots, k$ with Jacobian $y^{k-1}$. The joint density of $Z_1, \ldots, Z_{k-1}, Y$ is

$$f_{Z_1, \ldots, Z_{k-1}, Y}(z_1, \ldots, z_{k-1}, y|i) = \frac{1}{\Gamma(\alpha)^k \beta^k} \exp\left(-\frac{y}{\beta} \sum z_j\right) \left( \prod z_j \right)^{\alpha-1} \frac{1}{\lambda} \exp\left\{ \frac{y \bar{z}_i(\lambda - 1)}{\beta \lambda} \right\} y^{\lambda - 1}$$

Integrating out $y$ over $(0, \infty)$, we find the joint density of the maximal invariant to be

$$f_{Z_1, \ldots, Z_{k-1}}(z_1, \ldots, z_{k-1}|i) = \frac{1}{\lambda \Gamma(\alpha)^k \beta^k} \left( \prod z_j \right)^{\alpha - 1} \left( \sum z_j - \frac{z_i(\lambda - 1)}{\lambda} \right)^{-\lambda \alpha - 1}$$

for all $z_j > 0$.

(b) Out of the class of invariant rules satisfying conditions (a) and (b) of Theorem 1, the rule of (6.18) and (6.19) with $X_1, \ldots, X_k$ replaced by $Z_1, \ldots, Z_k$, maximizes the common value of $P(\text{accept } H_i|H_i)$. Here,

$$V = \max_i \frac{f_{Z_1, \ldots, Z_{k-1}}(Z_1, \ldots, Z_{k-1}|i)}{f_{Z_1, \ldots, Z_{k-1}}(Z_1, \ldots, Z_{k-1}|0)} = \max_i \left( 1 - \frac{z_i(\lambda - 1)}{\sum z_j \lambda} \right)^{-\lambda \alpha}$$

if and only if $\max_i (z_j/\sum z_j) > c'$ for some $c'$. Replacing $z_j$ by $x_j$ and letting $M = \max_i x_i/\sum x_j$, we find the optimal rule to be

$$\phi(0|x) = \begin{cases} 1 & \text{if } M \leq c' \\ 0 & \text{otherwise} \end{cases}$$
\[
\phi(i|x) = \begin{cases} 1 & \text{if } x_i / \sum x_j = M > c' \\ 0 & \text{otherwise} \end{cases}
\]

where \( c' \) is chosen so that condition (a) is satisfied. This rule is independent of \( \lambda \) provided \( \lambda > 1 \).

6.3.5. Assume that \( \Delta \) is known and \( \Delta > 0 \). The hypotheses are all simple and the invariant priors are of the form \( \tau_p(H_0) = 1 - pk(k - 1)/2 \), and \( \tau_p(H_{h,i}) = p \) for some \( 0 < p \leq 2/(k(k - 1)) \). Under \( H_0 \), the density of the observations is \( \prod_1^k \varphi(x_j) \) where \( \varphi(x) \) is the density of \( N(0,1) \). Under \( H_{h,i} \), this density is \( \left( \prod_1^k \varphi(x_j) \right) \left( \varphi(x_h - \Delta) / \varphi(x_h) \right) \left( \varphi(x_i - \Delta) / \varphi(x_i) \right) \). The ratio \( \varphi(x - \Delta) / \varphi(x) \) is equal to \( \exp \{ \Delta x - \Delta^2 / 2 \} \). Assuming zero/one loss, the Bayes rule reduces to the following.

\[
\phi(0|x) = \begin{cases} 1 & \text{if } \exp \{ M - \Delta^2 \} \leq (1/p) - (k(k - 1)/2) \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
\phi(h, i|x) = \begin{cases} 1 & \text{if } \exp \{ \Delta(x_h + x_i) - \Delta^2 \} = \exp \{ M - \Delta^2 \} > (1/p) - (k(k - 1)/2) \\ 0 & \text{otherwise} \end{cases}
\]

where \( M = \max_{h \neq i} \Delta(x_h + x_i) \). As in the proof of Theorem 1, we may replace the property of Bayesian optimality by Neyman-Pearson type optimality. Assume that \( \Delta > 0 \) and let \( M' = \max_{h \neq i} (x_h + x_i) \). Out of all rules satisfying (a) \( P(\text{accept } H_0|H_0) \geq 1 - \alpha \), and (b) \( P(\text{accept } H_{h,i}|H_{h,i}) \) is independent of \( h \) and \( i \), the rule,

\[
\phi(0|x) = \begin{cases} 1 & \text{if } M' \leq c \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
\phi(h, i|x) = \begin{cases} 1 & \text{if } (x_h + x_i) = M' > c \\ 0 & \text{otherwise} \end{cases}
\]

where \( c \) is chosen so that \( P(\text{accept } H_0|H_0) = 1 - \alpha \), maximizes the common value of \( P(\text{accept } H_{h,i}|H_{h,i}) \) subject to (a) and (b). This rule is independent of \( \Delta \) provided \( \Delta > 0 \).