Solutions to Exercises 6.1.1 through 6.1.3.

6.1.1. $f(x|\theta) = (\frac{5}{2})^x(1-\theta)^{5-x}$ has monotone likelihood ratio, so, if the loss function satisfies (6.3) with $\theta_1 = 1/3$ and $\theta_2 = 2/3$, we want to find one-sided $\phi$’s such that $E_{\theta_1}\phi_1(X) = E_{\theta_1}\psi_1(X)$ and $E_{\theta_2}\phi_2(X) = E_{\theta_2}\psi_2(X)$, where $\psi_j(x) = \sum_{i=0}^{x} \psi(i|x)$ for $j = 1, 2$. The rule that improves on $\psi(j|x)$ is then $\phi(j|x) = \phi_{j-1}(x) - \phi_j(x)$. Note that for a given monotone $\theta$, the monotone rules form an essentially complete class for this problem. Suppose that $\text{tanh}(a < 0)$, which holds if and only if $2\theta^2 = 72/243$. We need $40/243$ more; if we put $\phi_2(4) = 1$, we would get $80/243$ more. Therefore, the desired rule is: $\phi(3|x) = \phi(2|x) = 0$ for $x \leq 4$, $\phi(3|4) = 1/2$, and $\phi(3|5) = 1$. By symmetry, $\phi(1|x) = 0$ for $x \geq 2$, $\phi(1|1) = 1/2$, and $\phi(1|0) = 1$. We find $\phi(2|x)$ by subtracting the sum from 1: $\phi(2|0) = \phi(2|5) = 0$, $\phi(2|1) = \phi(2|4) = 1/2$, and $\phi(2|3) = \phi(2|4) = 1$.

6.1.2. (a) Consider the rule $\phi_a$ defined for $a \geq 0$ as

$$
\phi_a(1|x) = I_{(-\infty,-a)}(x),
\phi_a(2|x) = I_{(-a,a)}(x),
\phi_a(1|x) = I_{(a,\infty)}(x).
$$

This rule has risk function

$$
R(\theta, \phi_a) = (\theta + 1)^2P_\theta(X < -a) + \theta^2P_\theta(-a \leq X \leq a) + (\theta - 1)^2P_\theta(a < X)
$$

$$
= \theta^2 + (2\theta + 1)\Phi(-\theta - a) + (1 - 2\theta)(1 - \Phi(-\theta + a))
$$

where $\Phi(z)$ is the distribution function of $\mathcal{N}(0, 1)$. We now show that for each $\theta$ in $[-1,1]$ $R(\theta, \phi_a)$ is a decreasing function of $a$ for $a < .549 \cdots$. This shows that $\phi_a$ is not admissible for any $a < .549 \cdots$. The derivative of $R$ with respect to $a$,

$$
\frac{\partial}{\partial a} R(\theta, \phi_a) = -(2\theta + 1) \frac{1}{\sqrt{2\pi}} e^{-(\theta+a)^2/2} - (2\theta - 1) \frac{1}{\sqrt{2\pi}} e^{-(\theta-a)^2/2}
$$

is negative if and only if

$$
2\theta[e^{\theta} - e^{-\theta}] < [e^{\theta} + e^{-\theta}]
$$

which holds if and only if $2\theta\tanh(\theta/a) < 1$. This is symmetric in $\theta$ so we restrict $\theta$ to be in the interval $[0,1]$. Since $\tanh(z)$ is an increasing function, we may solve for $a$ to find $a < \tanh^{-1}(1/2\theta)/\theta$, where we think of $\tanh^{-1}(1/2\theta)$ as $+\infty$ if $\theta \leq 1/2$. But $\tanh^{-1}(1/2\theta)/\theta$ is otherwise decreasing in $\theta$ so the worst case occurs at $\theta = 1$. Hence, $R(\theta, \phi_a)$ is a decreasing function of $a$ for each $\theta$ in $[0,1]$ provided $a < \tanh^{-1}(1/2) = .549 \cdots$.

(b) Now suppose that $\Theta = (-\infty, \infty)$. To show that every monotone rule is admissible, it is sufficient to show that for a given monotone $\phi$ there does not exist a better monotone $\psi$, because Theorem 6.1.1 implies that the monotone rules form an essentially complete class for this problem. Suppose that

$$
\psi(1|x) = I_{(-\infty,a_1)}(x), \quad \psi(2|x) = I_{(a_1,b_1)}(x), \quad \psi(3|x) = I_{(b_1,\infty)}(x)
$$

$$
\phi(1|x) = I_{(-\infty,a_2)}(x), \quad \phi(2|x) = I_{(a_2,b_2)}(x), \quad \phi(3|x) = I_{(b_2,\infty)}(x)
$$

and that $\psi$ is as good as $\phi$, where $a_1 < b_1$ and $a_2 < b_2$. In terms of the risk functions, we have for all $\theta$,

$$
(\theta + 1)^2P_\theta(X < a_1) + \theta^2P_\theta(a_1 < X < b_1) + (1 - \theta)^2P_\theta(b_1 < X)
$$

$$
\leq (\theta + 1)^2P_\theta(X < a_2) + \theta^2P_\theta(a_2 < X < b_2) + (1 - \theta)^2P_\theta(b_2 < X)
$$
or, subtracting \( \theta^2 \) from both sides,

\[(2\theta + 1)P_\theta(X < a_1) + (1 - 2\theta)P_\theta(X > b_1) \leq (2\theta + 1)P_\theta(X < a_2) + (1 - 2\theta)P_\theta(X > b_2). \]

We use the fact that the normal tails decrease to zero very rapidly, namely, \( P_\theta(X > x) \sim (1/x)e^{-x^2/2}/\sqrt{2\pi} \). This follows from

\[xe^{-x^2/2}\int_0^\infty e^{-z^2/2}dz = x\int_0^\infty e^{-u(2u^2+2u)}^{-1/2}du = \int_0^\infty e^{-u(2u^2+2u)}^{-1/2}du \to 1\]
as \( x \to \infty \). Suppose that \( a_1 < a_2 \). Then subtract \( 2\theta + 1 \) from both sides of (*), divide both sides by \( P_\theta(X > a_1) \) and let \( \theta \to -\infty \). The left side tends to \( +\infty \) and the right side tends to zero. This contradiction shows that \( a_1 \geq a_2 \). Similarly, letting \( \theta \to +\infty \) shows that \( b_1 \leq b_2 \). Now suppose that \( a_2 < a_1 \). Then (*) implies that \( b_1 \neq b_2 \), and (*) reduces to

\[(2\theta + 1)P_\theta(a_2 < X < a_1) + (1 - 2\theta)P_\theta(b_1 < X < b_2) \leq 0.\]

This inequality is clearly false for any \( \theta \) such that \( P_\theta(a_2 < X < a_1) = P_\theta(b_1 < X < b_2) \). But such a \( \theta \) exists since both sides of this equation are continuous in \( \theta \) and the left side is less than (resp. greater than) the right for \( \theta \) sufficiently close to \( -\infty \) (resp. \( +\infty \)). Hence \( a_1 = a_2 \) and by symmetry, \( b_1 = b_2 \).

6.1.3. (a) We must find the value of \( x_2 \) that minimizes (6.10). The left side of the max in (6.10) is decreasing in \( x_2 \) for all \( \theta_0 \) and, if \( L = 1 \), the right side is increasing in \( x_2 \). This max is minimized therefore when \( x_2 \) is chosen so that the left side is equal to the right side, that is when \( P_{\theta_0}\{X < x_2\} + P_{\theta_0}\{X > x_2\} = P_{\theta_0}\{X < x_2\} \). For the logistic distribution, this is

\[(1 + e^{-(x_2-\theta_0)})^{-1} + 1 - (1 + e^{-(x_2-\theta_0)})^{-1} = (1 + e^{-(x_2-\theta_0)})^{-1}.\]

This reduces to \( 2e^{-x_2} + e^{\theta_0} = e^{x_2} \). For given \( \theta_0 \), this may be solved numerically for \( x_2 \). For example, if \( \theta_0 = 0.68437 \), \( x_2 = 1.00000 \), and if \( \theta_0 = 1.96268 \), then \( x_2 = 2.00000 \). (b) For \( L = 2 \), the minimum of (6.10) occurs when \( x_2 = \theta_0 \). (c) We will show the risk is made smaller if we increase \( x_2 \) slightly. The risk function of the rule (6.9) with \( x_1 = -x_2 \) as given above (6.10) becomes for the logistic distribution

\[R(\theta, \phi) = \begin{cases} 
(1 + e^{x_2+\theta})^{-1} + 1 - (1 + e^{x_2+\theta})^{-1} & \text{if } 0 \leq \theta \leq \theta_0 \\
(L-1)(1 + e^{x_2+\theta})^{-1} + (1 + e^{x_2+\theta})^{-1} & \text{if } \theta > \theta_0.
\end{cases}\]

For all \( \theta \leq \theta_0 \), the risk is decreasing in \( x_2 \). We complete the proof by showing that for all \( \theta > \theta_2 \), the derivative of \( R \) with respect to \( x_2 \) is negative if \( L \) is sufficiently large. For \( \theta > \theta_2 \), this derivative is

\[
\frac{\partial R}{\partial x_2} = \frac{(L-1)e^{x_2+\theta}}{(1 + e^{x_2+\theta})^2} + \frac{e^{-x_2+\theta}}{(1 + e^{-x_2+\theta})^2}.
\]

This is nonpositive if \( (L-1)e^{x_2+\theta}(1 + e^{-x_2+\theta})^2 \geq e^{-x_2+\theta}(1 + e^{x_2+\theta})^2 \), which holds if

\[(L-1)(e^{2x_2} + 2e^{x_2+\theta} + e^{2\theta}) \geq 1 + 2e^{x_2+\theta} + e^{2x_2+2\theta}.
\]

If \( L > 2 \), the first two terms on the left dominate the first two terms on the right, and if \( L-1 \geq e^{2x_2} \), the third term on the left also dominates the third term on the right for all \( \theta \), completing the proof. Note, however, that the inequality in the text, \( L-1 \geq e^{x_2} \), must be corrected to \( L-1 \geq e^{2x_2} \).