Solutions to Exercises 5.9.3 through 5.9.9.

5.9.3. \[ S^2 = \sum_{ij} \sum (X_{ij} - \xi - \mu_i - \eta_j)^2 \]
\[ = \sum_{ij} [(X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..}) + (\bar{X}_i - \bar{X}_.. - \mu_i) + (\bar{X}_j - \bar{X}_.. - \eta_j) - (\bar{X}_{..} - \xi)]^2 \]
\[ = \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})^2 + \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})^2 + \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})^2 + \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})^2 + \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})^2 + \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})^2 \]
\[ + \text{six cross product terms.} \]

We must show that all cross product terms are zero. This is illustrated for the first term,
\[ 2 \sum_{ij} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..})(\bar{X}_i - \bar{X}_.. - \mu_i) = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_i - \bar{X}_j - \bar{X}_{..}) = 0 \]

The inside summation over \( j \) is zero for all \( i \) because \( \sum_{j} X_{ij} = J \bar{X}_i \), and \( \sum_{j} \bar{X}_j = J \bar{X}_{..} \).

5.9.4. (a) We expand \( S^2 \) as follows.
\[ S^2 = \sum_{ijk} \sum (X_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \]
\[ = \sum_{ijk} \sum [(X_{ijk} - \bar{X}_{ijk}) - (\bar{X}_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})]^2 \]
\[ = \sum_{ijk} \sum (X_{ijk} - \bar{X}_{ijk})^2 + \sum_{ijk} \sum (\bar{X}_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \]
\[ + 2 \sum_{i} \sum_{j} (\bar{X}_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij}) (X_{ijk} - \bar{X}_{ijk}) \]

The last term is zero since \( \sum_{k} X_{ijk} = K \bar{X}_{ijk} \) for all \( i \) and \( j \). The middle term may be expanded into a sum of squares using the identity (5.128) (or Exercise 3) with \( X_{ij} \) replaced by \( \bar{X}_{ij} - \delta_{i,j} \).

(b) Using this identity, we find
\[ \min_{H} S^2 = \sum_{ijk} \sum (X_{ijk} - \bar{X}_{ijk})^2. \]

Similarly, putting \( \mu_i = 0 \) in the formula for part (a), we find
\[ \min_{H_0} S^2 = \sum_{ijk} \sum (X_{ijk} - \bar{X}_{ijk})^2 + \sum_{i} \sum_{j} (\bar{X}_{..} - \bar{X}_{..})^2. \]

The best invariant test rejects \( H_0 \) if
\[ F = \frac{\text{min}_{H_0} S^2 - \text{min}_{H} S^2}{\text{min}_{H} S^2/(n-k)} = \frac{\sum \sum (X_{..} - \bar{X}_{..})^2/(I-1)}{\sum \sum (X_{ijk} - \bar{X}_{ijk})^2/(IJ(K-1))} \]
is too large. Under \( H_0 \), this statistic has an \( F \)-distribution with \( I - 1 \) and \( IJ(K - 1) \) degrees of freedom. Under the general hypothesis, \( H \), it has a noncentral \( F \)-distribution with \( I - 1 \) and \( IJ(K - 1) \) degrees of freedom and noncentrality parameter \( \gamma^2 \), where \( \gamma^2 \) is computed as the numerator sum of squares divided by \( \sigma^2 \), with each \( X_{ijk} \) replaced by its expectation. This results in replacing \( \bar{X}_{i..} \) by \( \mu_i + \xi \) and \( \bar{X}_{..} \) by \( \xi \). We find
\[ \gamma^2 = \frac{1}{\sigma^2} \sum \sum \sum \mu_i^2 = \frac{1}{\sigma^2} JK \sum_{i} \mu_i^2. \]

(c) If we put \( \delta_{ij} = 0 \) in the identity of part (a), we find \( \min_{H_0} S^2 = \sum \sum (X_{ijk} - \bar{X}_{ijk})^2 + \sum \sum (X_{ij} - \bar{X}_i - \bar{X}_j + \bar{X}_{..})^2 \) so the best invariant test of no interaction effect rejects \( H_0 \) if
\[ F = \frac{\sum \sum (X_{ijk} - \bar{X}_{ijk} + \bar{X}_{..})^2/(IJ(K-1))}{\sum \sum (X_{ijk} - \bar{X}_{ijk})^2/(IJ(K-1))} \]
is too large. Under the general hypothesis, this has a noncentral $F$-distribution with $(I - 1)(J - 1)$ and $IJ(K - 1)$ degrees of freedom and noncentrality parameter

$$\gamma^2 = \frac{1}{\sigma^2} \sum \sum \delta_{ij}^2.$$  

5.9.5. (a) $S^2 = \sum (X_i - \beta_0 - \beta_1 z_i)^2$. We may find the least squares estimates of $\beta_0$ and $\beta_1$ by equating the partial derivatives of $S^2$ to zero.

$$\frac{\partial S^2}{\partial \beta_0} = -2 \sum (X_i - \beta_0 - \beta_1 z_i) = -2 \sum X_i - n\beta_0 = 0$$

since $\sum z_i = 0$, and

$$\frac{\partial S^2}{\partial \beta_1} = -2 \sum (X_i - \beta_0 - \beta_1 z_i)z_i = -2 \sum X_i z_i - \beta_1 \sum z_i^2 = 0.$$  

This gives $\hat{\beta}_0 = (1/n) \sum X_i$ and $\hat{\beta}_1 = \sum X_i z_i / \sum z_i^2$ as the least squares estimates.

(b) Under $H_0$, $S^2 = \sum (X_i - \beta_0)^2$ is minimized at $\hat{\beta}_0 = (1/n) \sum X_i = \bar{x}$.

(c) The UMP invariant test rejects $H_0$ if

$$F = \frac{\sum (\hat{\beta}_0 - \beta_0 - \hat{\beta}_1 z_i)^2/1}{\sum (X_i - \beta_0 - \beta_1 z_i)^2/(n - 2)} = \frac{(n - 2)\hat{\beta}_1^2}{\sum (X_i - \beta_0 - \beta_1 z_i)^2}$$

is too large. Under $H$, $F$ has a noncentral $F$-distribution with 1 and $n - 2$ degrees of freedom and noncentrality parameter, $\gamma^2 = (1/\sigma^2)(\sum (\beta_0 + \beta_1 z_i)z_i / \sum z_i^2)^2 = (1/\sigma^2)\beta_1^2$.

5.9.6. (a) The least squares estimates are the values of $\alpha$, $\beta$ and $\eta_i$ that minimize $S^2 = \sum \sum (X_{ij} - \alpha - \beta z_i - \eta_j)^2$ subject to $\sum_j \eta_j = 0$. We use Lagrange multipliers. The Lagrangian is $L = S^2 + \lambda \sum \eta_j$.

$$\frac{\partial L}{\partial \alpha} = -2 \sum \sum (X_{ij} - \alpha - \beta z_i - \eta_j) = -2 \sum \sum X_{ij} - IJ\alpha = 0$$

using $\sum z_i = 0$ and $\eta_j = 0$. This gives $\hat{\alpha} = \bar{X}_z$.

$$\frac{\partial L}{\partial \beta} = -2 \sum \sum (X_{ij} - \alpha - \beta z_i - \eta_j)z_i = -2 \sum \sum X_{ij}z_i - IJ\beta = 0$$

using $\sum z_i^2 = 1$. This gives $\hat{\beta} = (1/I) \sum z_i \bar{X}_z$.

$$\frac{\partial L}{\partial \eta_j} = -2 \sum_{i=1}^I (X_{ij} - \alpha - \beta z_i - \eta_j) + \lambda = -2I(\bar{X}_{.j} - \alpha - \eta_j) + \lambda = 0.$$  

We may find the Lagrange multiplier, $\lambda$, by summing over $j$. This gives $\lambda = 2I(\bar{X}_z)$. Substituting this value into the equation gives $\hat{\eta}_j = \bar{X}_{.j} - \bar{X}_z$ as the least squares estimate of $\eta_j$. Under $H_0$ the least squares estimates are easily found to be $\hat{\alpha} = \bar{X}_z$, $\hat{\beta} = (1/I) \sum z_i \bar{X}_z$, $\hat{\beta} = \bar{\beta}$. The UMP invariant test of $H_0$ rejects $H_0$ if

$$F = \frac{\sum \sum \hat{\eta}_j^2/(J - 1)}{\sum \sum (X_{ij} - \hat{\alpha} - \hat{\beta} z_i - \eta_j)^2/(IJ - J - 1)}$$

is too large.

(b) Under the general hypothesis, $F$ has a noncentral $F$-distribution with $J - 1$ and $IJ - J - 1$ degrees of freedom and noncentrality parameter, $\gamma^2 = (1/\sigma^2) \sum \sum \eta_j^2$. 

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5.9.7. Under the general linear hypothesis, (5.115), the log likelihood function of $\theta$ and $\sigma$ is

$$\log f(x|\theta, \sigma) = -n \log(\sqrt{2\pi\sigma}) - \frac{1}{2\sigma^2}(x - A\theta)^T(x - A\theta).$$

For each fixed $\sigma$, the maximum of $\log f$ over $\theta$ occurs at any value of $\theta$ that minimizes $(x - A\theta)^T(x - A\theta) = (x - \xi)^T(x - \xi)$. Since such values are independent of $\sigma$, any maximum likelihood estimate of $\theta$ is also a least squares estimate of $\theta$ and conversely. Therefore, if $\hat{\theta}$ denotes a maximum likelihood estimate, $\hat{\xi} = A\hat{\theta}$ is both the maximum likelihood estimate and the least squares estimate of $\xi$.

5.9.8. We have $r = I - 1$, $k = I$ and $n = n_1 + \cdots + n_J$. The distribution of the $F$-statistic under the general hypothesis is noncentral $F$-distribution, $F_{r,n-k}(\gamma^2) = F_{I-1,n-I}(\gamma^2)$. To find $\gamma^2$, we use (5.126). The numerator sum of squares is

$$\sum_i \sum_j (\bar{X}_{i.} - \bar{X}_{..})^2 = \sum_i n_i(\bar{X}_{i.} - \bar{X}_{..})^2.$$  

Replacing each $X_{ij}$ in this by its expectation $\bar{X}_{i.}$, we obtain

$$\gamma^2 = (1/\sigma^2) \sum_i n_i(\bar{X}_{i.} - \bar{\theta})^2.$$  

5.9.9. (a) $S^2 = \sum \sum \sum (X_{ijk} - \xi - \lambda_i - \mu_j - \eta_k)^2 = \sum \sum \sum (X_{ijk} - \bar{X}_{..} - \bar{X}_{.j} - \bar{X}_{.k} + 2\bar{X}_{..})^2 + \sum \sum \sum (\bar{X}_{i..} - \bar{X}_{..} - \lambda_i)^2 + \sum \sum \sum (\bar{X}_{.j} - \bar{X}_{..} - \mu_j)^2 + \sum \sum \sum (\bar{X}_{.k} - \bar{X}_{..} - \eta_k)^2 + \sum \sum \sum (\bar{X}_{..} - \xi)^2.$

(b) Under $H$, the least squares estimates are $\hat{\xi} = \bar{X}_{..}$, $\hat{\lambda}_i = \bar{X}_{i..} - \bar{X}_{..}$, $\hat{\mu}_j = \bar{X}_{.j} - \bar{X}_{..}$, and $\hat{\eta}_k = \bar{X}_{.k} - \bar{X}_{..}$. Under $H_0$, they are the same except that $\hat{\lambda}_i = 0$. So the best invariant test of $H_0$ rejects $H_0$ when

$$F = \frac{\sum \sum \sum (\bar{X}_{i..} - \bar{X}_{..})^2 / (I - 1)}{\sum \sum \sum (X_{ijk} - \bar{X}_{i..} - \bar{X}_{.j} - \bar{X}_{.k} + 2\bar{X}_{..})^2 / (IJK - I - J - K + 2)}$$

is too large. Under $H$, this has a noncentral $F$-distribution with $I - 1$ and $IJK - I - J - K + 2$ degrees of freedom and noncentrality parameter $\gamma^2 = (1/\sigma^2) \sum \sum \sum \lambda_i^2$. 

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