Solutions to the Exercises of Section 5.1.

5.1.1. Let us use the notation, \( \alpha_0(\phi) = R(\theta_0, \phi) = E_{\theta_0}(\phi(X)) \) and \( \alpha_1(\phi) = R(\theta_1, \phi) = 1 - E_{\theta_1}(\phi(X)) \). We are given that \( \phi \) is not admissible. This means that there is a test \( \phi' \) better than \( \phi \), which means that both \( \alpha_0(\phi') \leq \alpha_0(\phi) \) and \( \alpha_1(\phi') \leq \alpha_1(\phi) \) with at least one a strict inequality. But since \( \phi \) is best of size \( \alpha_0 \) and \( \alpha_0(\phi') \leq \alpha_0(\phi) \), we must have \( \alpha_1(\phi') \geq \alpha_1(\phi) \). Hence, \( \alpha_1(\phi') = \alpha_1(\phi) \), and therefore, \( \alpha_0(\phi') < \alpha_0(\phi) \). We are to show that \( \alpha_1(\phi) \) cannot be positive.

If \( \alpha_1(\phi) > 0 \), define a new test \( \phi'' = \lambda \phi' + 1 - \lambda \), where \( \lambda \) is chosen so that \( \alpha_0(\phi'') = \alpha_0(\phi) \) i.e. \( \lambda \alpha_0(\phi') + 1 - \lambda = \alpha_0(\phi) \). This gives \( \lambda = (1 - \alpha_0(\phi)) / (\alpha_0(\phi) - \alpha_0(\phi')) \), so that \( 0 \leq \lambda < 1 \). But then

\[
\alpha_1(\phi'') = 1 - E_{\theta_0}(\phi'(X)) - 1 + \lambda = \lambda \alpha_1(\phi') = \lambda \alpha_1(\phi) < \alpha_1(\phi).
\]

Thus, if \( \alpha_1(\phi) > 0 \) we have \( \alpha_1(\phi'') < \alpha_1(\phi) \) which contradicts the assumption that \( \phi \) is a best test of size \( \alpha_0 \).

5.1.2. Let \( \phi_0 \) be of the given form and let \( 0 \leq \phi \leq 1 \) be any other function. Then,

\[
\int (\phi_0(x) - \phi(x))(f_0(x) - \sum_{j=1}^{n} k_j f_j(x)) \, dx \geq 0,
\]

since the integrand is nonnegative from the definition of \( \phi_0 \). Hence,

\[
0 \leq \int \phi_0(x)f_0(x) \, dx - \int \phi(x)f_0(x) \, dx - \sum_{j=1}^{n} k_j \int (\phi_0(x) - \phi(x))f_j(x) \, dx.
\]

If

\[
\int \phi(x)f_j(x) \, dx = \int \phi_0(x)f_j(x) \, dx \quad \text{for} \quad j = 1, \ldots, n,
\]

then each term of the summation in (1) is zero so that

\[
\int \phi(x)f_j(x) \, dx \leq \int \phi_0(x)f_j(x) \, dx \quad \text{as was to be shown. If} \quad k_j \geq 0 \quad \text{for all} \quad j,
\]

and if

\[
\int \phi(x)f_j(x) \, dx \leq \int \phi_0(x)f_j(x) \, dx \quad \text{for} \quad j = 1, \ldots, n,
\]

then each term of the summation in (1) is nonnegative, so we still have (2).

5.1.3. The likelihood ratio, \( f_1(x)/f_0(x) \) takes on three values 0, 1 and \( \infty \), with probabilities 1/2, 1/2 and 0 under \( H_0 \) and probabilities 0, 1/2, and 1/2 under \( H_1 \). Rejecting \( H_0 \) if \( X > 1 \) gives the point \( (0,1/2) \) in the risk set. Rejecting \( H_0 \) if \( 1/2 < X < 1 \) gives the point \( (1/2,0) \) in the risk set. This gives the lower boundary of the risk set \( S \) to be the lines from \( (0,1/2) \) to \( (1/2,0) \). The risk set is given in Figure 1.

5.1.4. The likelihood ratio, \( f_1(x)/f_0(x) \) takes on the three values 4/9, 8/9 and 16/9, with probabilities 1/4, 1/2 and 1/4 under \( H_0 \) and probabilities 1/9, 4/9, and 4/9 under \( H_1 \). Rejecting \( H_0 \) if \( X = 2 \) gives the risk point \( (1/4,5/9) \), and rejecting \( H_0 \) if \( X \geq 1 \) gives the point \( (3/4,1/9) \) in the risk set. The complete risk set is given in Figure 2.

5.1.5. The likelihood ratio is \( f_1(x)/f_0(x) = e^{x^2/2}/2 \). The best tests reject \( H_0 \) when this ratio is greater than some constant, or equivalently, when \( X \) is greater than some constant, say \( X > c \). The probability of error type I is \( \alpha_0 = P_0(X > c) = e^{-c} \). The probability of error type II is \( \alpha_1 = P_1(X < c) = 1 - e^{-c}/2 \). The lower boundary of the risk set therefore satisfies \( \alpha_1 = 1 - \sqrt{\alpha_0} \). The complete risk set is given in Figure 3.
5.1.6. Since \( f(x) > 0 \) for all \( x \), the best tests have the form: \( \phi(x) \) is 1, is arbitrary, or is 0 according as the likelihood ratio \( \lambda(x) = f_1(x)/f_0(x) \) is greater than \( k \), equal to \( k \), or less than \( k \). In the case where \( X \) is \( \mathcal{C}(\theta, 1) \) and \( \theta_0 = 0 \) and \( \theta_1 = 1 \), the likelihood ratio is

\[
\lambda(x) = \frac{(1 + x^2)}{(1 + (x - 1)^2)}
\]

By evaluating the derivative \( \lambda' \), it may be seen that \( \lambda(x) \) starts at 1 at \( x = -\infty \), decreases to a minimum at \( x = (1 - \sqrt{5})/2 = -0.618 \ldots \), increases to a maximum at \( x = (1 + \sqrt{5})/2 = 1.618 \ldots \), and then decreases to 1 as \( x \) tends to \( \infty \). Hence, since \( \lambda(1) = \lambda(3) = 2 \), the interval \((1, 3)\) is a best test of its size, corresponding to \( k = 2 \). The power function is \( \beta(\theta) = P_0(1 < X < 3) \), is symmetric in \( \theta \) about \( \theta = 2 \), attains its maximum value of 1/2 at \( \theta = 2 \), and decreases to 0 as \( \theta \to \infty \). Other values are \( \beta(1) = .352 \ldots \), \( \beta(0) = .148 \ldots \), and \( \beta(-1) = .070 \ldots \).

5.1.7. If \( \phi \) is a best test of size \( \alpha \) and \( E_{\theta_i}\phi(X) = \alpha \), then \( \phi_1(x) \equiv \alpha \) is also a best test of size \( \alpha \) since it has the same power as \( \phi \). But from the unicity part of the Neyman-Pearson Lemma with \( \alpha > 0 \), \( \phi_1 \) must have the form (5.7). But since 0 < \( \alpha < 1 \), this implies \( f_1(x) = k f_0(x) \) a.s. for some \( k \geq 0 \). And since both \( f_1 \) and \( f_0 \) are densities, \( k \) must be equal to 1. This implies \( P_{\theta_0} = P_{\theta_1} \).

5.1.8. The best tests of the form (5.7) become

\[
\phi(z) = \begin{cases} 
1 & \text{if } \prod \frac{\theta_i^{x_i}}{\theta_i^{0x_i}} > k \prod \frac{\theta_i^{0x_i}}{\theta_i^{x_i}} \\
\gamma(x) & \text{if } \prod \frac{\theta_i^{x_i}}{\theta_i^{0x_i}} = k \prod \frac{\theta_i^{0x_i}}{\theta_i^{x_i}} \\
0 & \text{if } \prod \frac{\theta_i^{x_i}}{\theta_i^{0x_i}} < k \prod \frac{\theta_i^{0x_i}}{\theta_i^{x_i}} 
\end{cases}
\]

where \( r_i = \theta_i^x/\theta_i^0 \) and \( k' = \log k \). The test of the form (5.8) becomes

\[
\phi(z) = \begin{cases} 
1 & \text{if } z_i > 0 \text{ for some } \theta_i^0 = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

In the special case \( k = 4 \), \( r_1 = .10/1.55 \), \( r_2 = .40/2.0 \), \( r_3 = .30/1.5 \), and \( r_4 = .20/1.0 \), the tests reject \( H_0 \) for large values of \( Z_1 \log(2/11) + (Z_2 + Z_3 + Z_4) \log 2 \). But since \( Z_1 = n - (Z_2 + Z_3 + Z_4) \), the tests reject \( H_0 \) for large values of \( Z_2 + Z_3 + Z_4 \). Under \( H_0 \), \( Z_2 + Z_3 + Z_4 \in \mathcal{B}(n, .45) \), and under \( H_1 \), \( Z_2 + Z_3 + Z_4 \in \mathcal{B}(n, .90) \). The error probabilities may be computed from this.

5.1.9. Note that \( \int \phi_0(x) f_0(x) \, dx = \int f_0(x)^+ \, dx \). The only functions \( \phi(x) \), \( 0 \leq \phi(x) \leq 1 \), that satisfy \( \int \phi(x) f_0(x) \, dx = \int f_0(x)^+ \, dx \) are the functions

\[
\phi(x) = \begin{cases} 
1 & \text{if } f_0(x) > 0 \\
\gamma(x) & \text{if } f_0(x) = 0 \\
0 & \text{if } f_0(x) < 0
\end{cases}
\]

Therefore to maximize \( \int \phi(x) f_1(x) \, dx \) out of this class, we may take \( \gamma(x) \) to be of the form

\[
\gamma(x) = \begin{cases} 
1 & \text{if } f_0(x) = 0 \text{ and } f_1(x) > 0 \\
\text{any} & \text{if } f_0(x) = 0 \text{ and } f_1(x) = 0 \\
0 & \text{if } f_0(x) = 0 \text{ and } f_1(x) < 0
\end{cases}
\]

The given \( \phi_0(x) \) has \( \gamma(x) \) of this form.