Solutions to the Exercises of Section 2.1.

2.1.1. Proof. Let \( C \) be a complete class of decision rules, and let \( A \) denote the class of admissible rules. We are to show \( A \subset C \). If \( \delta \not\in C \), then there is a \( \delta_0 \in C \) that is better than \( \delta \). This shows that \( \delta \) is not admissible, i.e., \( \delta \not\in A \). So, \( C^c \subset A^c \), or \( A \subset C \). (\( A^c \) denotes the complement of the set \( A \).)

2.1.2. Proof. Let \( C \) be essentially complete and let \( \delta \) be admissible. If \( \delta \not\in C \), then, using the essential completeness of \( C \), there exists a \( \delta_0 \in C \) such that \( \delta_0 \) is as good as \( \delta \). But since \( \delta \) is admissible, \( \delta_0 \) cannot be better than \( \delta \). Thus \( \delta_0 \) and \( \delta \) have the same risk function and are equivalent.

2.1.3. If \( C \) is a complete class, then for every rule \( \delta \) not in \( C \), there exists a rule \( \delta' \) in \( C \) such that \( \delta' \) is better than \( \delta \). But if \( \delta' \) is better than \( \delta \), then \( \delta' \) is also as good as \( \delta \), so \( C \) satisfies the definition of being an essentially complete class.

2.1.4. Let \( C \) be a minimal complete class. We may break \( C \) up into equivalence classes, \( C = \cup_{\alpha \in A} C_\alpha \), where for each \( \alpha \), any two elements of \( C_\alpha \) have the same risk function, and if \( \alpha \neq \beta \) then \( C_\alpha \cap C_\beta = \emptyset \) and elements of \( C_\alpha \) and \( C_\beta \) have different risk functions. Then by the axiom of choice, there is a set \( D \) consisting of one point from each of the sets \( C_\alpha \) for \( \alpha \in A \). Then the set \( D \) is minimal essentially complete.

2.1.5. The proof uses the result of Exercise 2.1.3, that every complete class is essentially complete. Suppose that \( C \) is a complete class and that \( C \) contains no proper subclass that is essentially complete. Then it can contain no proper subclass that is complete either, so \( C \) is minimal complete. Moreover, \( C \) is essentially complete, and so is minimal essentially complete.

2.1.6. Proof. Let \( A \) denote the class of admissible rules and suppose that \( A \) is complete. We must show that no proper subclass of \( A \) is complete. Let \( A' \) be a proper subclass of \( A \) so that \( A - A' \) is nonempty. Let \( \delta \in A - A' \). If \( A' \) were complete, then there would be a rule \( \delta_0 \in A' \) that is better than \( \delta \). But then \( \delta \) would not be admissible and so would not be in \( A \). This contradiction completes the proof.

2.1.7. Suppose that \( \Theta \) consists of one point, say \( \Theta = \{0\} \), and that the risk set is the open interval, \( S = (0,1) \). (Such a situation occurs if \( A = (0,1) \) and \( L(0,a) = a \) for all \( a \in A \).) If \( C_1 \) denotes the class of decision rules with rational risk points, and if \( C_2 \) denotes the complement of \( C_1 \), then both \( C_1 \) and \( C_2 \) are complete, but \( C_1 \cap C_2 \) is empty and so is not essentially complete.