Large Sample Theory
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Exercises, Section 23, Minimum Chi-Square Estimates.

1. Let $X_1, \ldots, X_n$ be a sample from the $G(1, \theta)$ distribution, with density $f(x) = \theta^{-1} e^{-x/\theta} I(x > 0)$, and let $Y_1, \ldots, Y_n$ be an independent sample from the $G(1, \theta^2)$ distribution, where $\theta > 0$ is an unknown parameter.
   
   (a) Find the $\chi^2, Q_n(\pi(\theta))$ of Example 1, where $Z_n = (X_n, Y_n)^T$.
   
   (b) Find the transformed $\chi^2$ with the transformation $g(x, y) = (x, \sqrt{y})$.
   
   (c) Find the minimum modified, transformed $\chi^2$ estimate of $\theta$. (Ans. $\hat{\theta}_n = (\bar{X}_n + 4\sqrt{\bar{Y}_n})/5$.) Compare to the maximum likelihood estimate. Which is better asymptotically?

2. Let $X = (X_1, \ldots, X_c)$ have a multinomial distribution with sample size $n = 1$ and probability vector $p(\theta) = (p_1(\theta), \ldots, p_c(\theta))^T > 0$, where $1^T p(\theta) = \sum_1^c p_i(\theta) = 1$ for all $\theta = (\theta_1, \ldots, \theta_k)^T \in \Theta$, an open set in $k$-dimention, $k < c$. Assume $\dot{p}(\theta)$ (a $c$ by $k$ matrix) exists for all $\theta \in \Theta$, and note that $1^T \dot{p}(\theta) = 0$. Show that Fisher Information is $I(\theta) = \dot{p}(\theta)^T P(\theta)^{-1} \dot{p}(\theta)$, where $P(\theta) = \text{diag}(p(\theta))$. 