Exercises, Section 14, Asymptotic Theory of Extreme Order Statistics.

1. Let $X_1,\ldots,X_n$ be a sample from a distribution with distribution function $F(x)$ and let $M_n = \max\{X_1,\ldots,X_n\}$ be the maximum of the sample. Find a normalization, $(M_n - a_n)/b_n$, if any exists, that has a nondegenerate limiting distribution as $n \to \infty$, for the following distributions:

   (a) The logistic distribution with density $f(x) = e^{-x}/(1 + e^{-x})^2$.
   
   (b) The distribution with distribution function $F(x) = 1 - (\log(x + 1))/x$ for $x > 0$.
   
   (c) The cosine distribution with density $f(x) = (1/2)\cos(x)I(-\pi/2 < x < \pi/2)$.

2. Suppose $X$ has the $G_{1,\gamma}(x)$ distribution, and let $Y = \gamma(X - 1)$. Show that as $\gamma \to \infty$, $Y$ converges in law to the $G_3$ distribution.

3. Suppose in Example 6 that $F(x) = \Phi(x - \mu)$ so that we are sampling from a normal distribution with mean $\mu$.

   (a) Find the asymptotic distribution of $M_n$.
   
   (b) Show that $\hat{\mu}_n \defeq M_n - \sqrt{2\log n} \overset{P}{\longrightarrow} \mu$.

   Thus $\hat{\mu}_n$ is a consistent estimate of $\mu$. What is its asymptotic efficiency relative to $\overline{X}_n$?

4. Let $X_1,\ldots,X_n$ be a sample from the distribution on the interval $(-\pi/2,0)$, with distribution function $F(x) = \cos(x)$ for $-\pi/2 < x < 0$, and let $M_n$ represent the maximum of the sample.

   (a) What is the distribution function of $M_n$?
   
   (b) Find $b_n$ such that $M_n/b_n$ converges in law to a nondegenerate distribution and find the distribution.

5. Let $X_1, X_2,\ldots$ be i.i.d. from a distribution with density $f(x)$ such that $f(x) = 0$ for $x < a$, $f(a) > 0$ and $f(x)$ is right continuous at $a$. Show that $n\min\{X_1,\ldots,X_n\} - a$ converges in law to the exponential distribution with rate parameter $f(a)$ (i.e. mean $1/f(a)$).

6. What can you say about the asymptotic distribution of $\min\{X_1,\ldots,X_n\}$ when $X_1,\ldots,X_n$ are i.i.d. with distribution $G_3$? What is the exact distribution?