Exercises, Section 6, Slutsky Theorems.

1. (G. Blom) An urn contains one white and one black ball. Draw a ball at random. With probability 1/2, return it to the urn; otherwise (again with probability 1/2) put a ball of the opposite color in the urn. Perform $n$ such drawings in succession. Find the limiting distribution of $(X_n - \operatorname{EX}_n)/\sqrt{n}$, where $X_n$ is the number of white balls appearing in the $n$ draws.

2. Let $X_1, X_2, \ldots$ be i.i.d. double exponential (Laplace) random variables with density, $f(x) = (2\tau)^{-1} \exp(-|x|/\tau)$, where $\tau$ is a positive parameter that represents the mean deviation, $\tau = \operatorname{E}\{|X|\}$. Let $\overline{X}_n = n^{-1} \sum_1^n X_i$ and $\overline{Y}_n = n^{-1} \sum_1^n |X_i|$.

(a) Find the joint asymptotic distribution of $\overline{X}_n$ and $\overline{Y}_n$.
(b) Find the asymptotic distribution of $(\overline{Y}_n - \tau)/\overline{X}_n$.

3. Suppose

$$\sqrt{n} \begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} \rightarrow \mathcal{N} \begin{pmatrix} U \\ V \\ W \end{pmatrix} \begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \end{pmatrix}$$

Show using Slutsky’s Theorem that

$$\sqrt{n}(X_n + Y_n Z_n - \mu_x - \mu_y \mu_z) \overset{\mathcal{L}}{\rightarrow} \mathcal{N}(0, \sigma^2)$$

for some $\sigma^2$, and find $\sigma^2$.

4. Suppose $X_1, \ldots, X_n$ is a sample from the uniform distribution on the interval $(0, \theta)$, $\mathcal{U}(0, \theta)$. The maximum likelihood estimate of $\theta$ is $M_n$, the maximum of the sample. In Chapter 14, we will see that $n(\theta - M_n) \overset{\mathcal{L}}{\rightarrow} Z$, where $Z$ has the exponential distribution, $\mathcal{G}(1, \theta)$. As an estimate of $\theta$, $M_n$ might not be so good since $M_n < \theta$ with probability 1, but we might use $((n + c)/n)M_n$ for some positive number $c$.

(a) What is the asymptotic distribution of $((n + c)/n)M_n$?
(b) What value of $c$ should be used if we measure the accuracy of the estimate by squared error loss?
(c) What value of $c$ should be used if we measure the accuracy of the estimate by absolute error loss?

5. Suppose $X_n$ has a binomial distribution with sample size, $n$, and probability of success, $p$. Let $Y_n = \max\{X_n/n, 1 - X_n/n\}$. What is the asymptotic distribution of $Y_n$,

(a) when $p \neq 1/2$?
(b) when $p = 1/2$?

6. We say a sequence of random variables, $X_n$, is tight or bounded in probability, if for every $\epsilon > 0$, there exists a number $M$ such that for all $n$, $P(|X_n| > M) < \epsilon$. Show that if $X_n$ is tight and $Y_n \overset{P}{\rightarrow} 0$, then $X_n Y_n \overset{P}{\rightarrow} 0$. 

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Large Sample Theory
Ferguson