Solutions 1

(15) 1. (a) Find the average value of the function \( f(x) = x^2 \sqrt{1 + x^3} \) on the interval \([0,2]\).

\[
\text{Average value} = \frac{1}{2} \int_0^2 x^2 \sqrt{1 + x^3} \, dx = \frac{1}{6} \int_1^9 \sqrt{u} \, du = \frac{26}{9}.
\]

(b) Give a precise statement of the Mean Value Theorem for Integrals. See page 403 of the text.

(c) If \( g \) is continuous and \( \int_1^3 g(x) \, dx = 8 \), explain why \( g \) must take the value 4 at least once in the interval \([1,3]\).

By the Mean Value Theorem for Integrals, there is a \( c \) in \([1,3]\) so that \( 8 = \int_1^3 g(x) \, dx = 2g(c) \). For this \( c \), \( g(c) = 4 \).

(15) 2. Find the number \( a \) so that the line \( x = a \) bisects the area under the curve \( y = \frac{1}{x^2}, 1 \leq x \leq 4 \).

The area corresponding to \( 1 \leq x \leq a \) is

\[
\int_1^a \frac{1}{x^2} \, dx = \frac{a - 1}{a}.
\]

Therefore the \( a \) we want is the solution of \( (a - 1)/a = (1/2)(3/4) \), or \( a = 8/5 \).

(15) 3. Use slices to find the volume of the solid obtained by rotating the region bounded by \( y = x, y = \sqrt{x} \) about the line \( x = 2 \).

\[
\text{Volume} = \pi \int_0^1 \left( (2 - y^2)^2 - (2 - y)^2 \right) \, dy = \frac{8}{15}\pi.
\]

(15) 4. Use cylindrical shells to compute the volume of the solid generated by rotating the region bounded by \( y = x^2, y = 0, x = 1, x = 2 \) about the line \( x = 1 \).

\[
2\pi \int_1^2 (x - 1)x^2 \, dx = \frac{17}{6}\pi.
\]

(15) 5. (a) Find a formula for the inverse of the function

\[
f(x) = \frac{4x - 1}{2x + 3}.
\]

Solving

\[
y = \frac{4x - 1}{2x + 3}
\]

for \( x \) gives

\[
x = f^{-1}(y) = \frac{1 + 3y}{4 - 2y}.
\]
(b) Suppose $g$ is the inverse function of $f$, and $f(4) = 5, f'(4) = 2/3$. Find $g'(5)$.

\[ g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(4)} = \frac{3}{2}. \]

(10) 6. Find the equation of the tangent line to $y = e^x/x$ at the point $(1, e)$.

\[ y' = \frac{e^x(x-1)}{x^2}, \]

which is 0 at $x = 1$. Therefore, the tangent line has equation $y = e$.

(15) 7. Approximate $\int_1^5 x^2 \, dx$ using $n = 2$ in

(a) the midpoint rule.

\[ 2(4 + 16) = 40. \]

(b) the trapezoidal rule.

\[ 1 + 18 + 25 = 44. \]

(c) Simpson’s rule.

\[ \frac{2}{3}(1 + 36 + 25) = \frac{124}{3}. \]

**Solutions 2**

(15) 1. (a) Find the average value of the function $f(t) = t\sqrt{1 + t^2}$ on the interval $[0,5]$.

Average value = $\frac{1}{5} \int_0^5 t\sqrt{1 + t^2} \, dt = \frac{1}{10} \int_1^{26} \sqrt{u} \, du = \frac{26^{3/2} - 1}{15}$.

(b) Give a precise statement of the Mean Value Theorem for Integrals. See page 403 of the text.

(c) If $g$ is continuous and $\int_1^3 g(x) \, dx = 8$, explain why $g$ must take the value 4 at least once in the interval $[1,3]$.

By the Mean Value Theorem for Integrals, there is a $c$ in $[1,3]$ so that $8 = \int_1^3 g(x) \, dx = 2g(c)$. For this $c$, $g(c) = 4$.

(15) 2. Find the number $b$ so that the line $y = b$ bisects the area under the curve $y = 1/x^2, 1 \leq x \leq 4$.

The area corresponding to $b \leq y \leq 1$ (if $\frac{1}{16} < b < 1$) is

\[ \int_b^1 \left( \frac{1}{\sqrt{y}} - 1 \right) \, dy = 1 - 2\sqrt{b} + b = (1 - \sqrt{b})^2. \]
Therefore the $b$ we want is the solution of \((1 - \sqrt{b})^2 = (1/2)(3/4)\), or \(b = (1 - \sqrt{3/8})^2\).

(15) 3. Use slices to find the volume of the solid obtained by rotating the region bounded by \(y = x\) and \(y = \sqrt{x}\) about the line \(x = -1\).

\[
\text{Volume} = \pi \int_{0}^{1} ((1 + y)^2 - (1 + y^2)^2) \, dy = \frac{7}{15} \pi.
\]

(15) 4. Use cylindrical shells to compute the volume of the solid generated by rotating the region bounded by \(y = x^2, y = 0, x = 1, x = 2\) about the line \(x = 2\).

\[
2\pi \int_{1}^{2} (2 - x)x^2 \, dx = \frac{11}{6} \pi.
\]

(15) 5. (a) Find a formula for the inverse of the function

\[
f(x) = \frac{4x - 1}{3x + 2}.
\]

Solving

\[
y = \frac{4x - 1}{3x + 2}
\]

for \(x\) gives

\[
x = f^{-1}(y) = \frac{1 + 2y}{4 - 3y}.
\]

(b) Suppose \(g\) is the inverse function of \(f\), and \(f(4) = 5, f'(4) = 3/2\). Find \(g'(5)\).

\[
g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(4)} = \frac{2}{3}.
\]

(10) 6. Find the equation of the tangent line to \(y = e^x/x\) at the point \((1, e)\).

\[
y' = \frac{e^x(x - 1)}{x^2},
\]

which is 0 at \(x = 1\). Therefore, the tangent line has equation \(y = e\).

(15) 7. Approximate \(\int_{0}^{4} x^2 \, dx\) using \(n = 2\) in

(a) the midpoint rule.

\[
2(1 + 9) = 20.
\]

(b) the trapezoidal rule.

\[
0 + 8 + 16 = 24.
\]

(c) Simpson's rule.

\[
\frac{2}{3}(0 + 16 + 16) = \frac{64}{3}.
\]