A Proof of Carleson’s theorem

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Carleson’s Theorem

For each Schwartz function $f \in S(\mathbb{R})$ define

$$Cf(x) := \sup_N \left| \int_{-\infty}^{N} \hat{f}(\xi)e^{2\pi i \xi x} \, d\xi \right|,$$

where the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) := \int f(x)e^{-2\pi i \xi x} \, dx.$$  

Then $C$ is of weak type $2,2$:

**Theorem (Carleson, 1966)**

There is a $C > 0$ such that for all $f \in S(\mathbb{R})$

$$\sup_{\lambda > 0} \lambda \left| \{x : Cf(x) > \lambda\} \right|^\frac{1}{2} \leq C\|f\|_2.$$
Wave Packets

Define translation, modulation, dilation by:

\[ T_y f(x) = f(x - y) \]
\[ M_\eta f(x) = e^{2\pi i \eta x} f(x) \]
\[ D_\Lambda^p f(x) = \Lambda^{-p} f(\Lambda^{-1} x) \]

Pick \( \phi \in \mathcal{S}(\mathbb{R}) \) so that \( \hat{\phi} \) looks like

Define the wave packet \( \phi_{1P} \) associated to a rectangle \( P = I_P \times \omega_P \) of area 1 by

\[ \phi_{1P} = M_{c(\omega_{1P})} T_c(I_P) D_{|I_P|} \phi \]
The Discrete Hilbert Transform

A *tile* is a rectangle of area 1 of the form

\[ [2^k n, 2^k(n + 1)] \times [2^{-k} l, 2^{-k}(l + 1)] \]

with integers \( k, n, l \). Let \( \overline{P} \) be the set of tiles.

Define for \( \xi \in \mathbb{R} \)

\[
A_\xi f = \sum_{P \in \overline{P}} 1_{\omega_2 P}(\xi) \langle f, \phi_{1P} \rangle \phi_{1P}
\]

This is nonzero, positive semidefinite, and vanishes if \( \hat{f} \) is supported in \([\xi, \infty]\).

\[
\begin{array}{l}
\xi \\
\hline
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\hline
\end{array}
\]

Observe that for integers \( k \) we have

\[
A_\xi = D_{2^{-k}}^2 A_{2^{-k}\xi} D_{2^k}^2
\]
The Hilbert transform as average

Define $H_\xi$ by the average

$$\lim_{N \to \infty} \int M_{-\eta} T_{-y} D_{2-\kappa}^2 A_{2-\kappa}(\eta + \xi) D_{2\kappa}^2 T_y M_\eta$$

Where the average is over the set of all

$$(\eta, y, \kappa) \in [-N, N] \times [-N, N] \times [0, 1]$$

Again this nonzero and vanishes if $\hat{f}$ is supported in $[\xi, \infty]$.

Moreover it commutes with translations $T_z$, $z \in \mathbb{R}$ and with dilations $M_\xi D_\lambda^2 M_{-\xi}$, $\lambda > 0$ about the point $\xi$. Hence

$$H_\xi f(x) = c_\xi \int_{-\infty}^{\xi} \hat{f}(\eta) e^{2\pi i x \eta} \, d\eta$$

Conjugation with $M_\eta$ gives that $c_\xi$ indeed is independent of $\xi$. 
The Discrete Carleson Theorem

We will prove that

\[ \left\| \sup_\xi |A_\xi f| \right\|_{2,\infty} \leq C \|f\|_2 \]

Then by averaging:

\[ \left\| \sup_\xi |H_\xi f| \right\|_{2,\infty} \leq C \|f\|_2 \]

The first estimate follows from

\[ \sum_{P \in \overline{P}} \left| \langle f, \phi_{1P} \rangle \langle \phi_{1P} \cdot (1_{\omega_{2P} \circ N}) \cdot 1_E \rangle \right| \leq C \|f\|_2 |E|^{1/2} \]

for all \( f \in S(\mathbb{R}) \), measurable sets \( E \), and measurable functions \( N \). We can assume \( \overline{P} \) is finite, \( \|f\| = 1 \), \( |E| = 1 \). Thus we have to show

\[ \sum_{P \in \overline{P}} \left| \langle f, \phi_{1P} \rangle \langle \phi_{1P}, 1_{E_{2P}} \rangle \right| \leq C \]

where

\[ E_{2P} = E \cap \{x : N(x) \in \omega_P\} \]
Trees

Define $P < P'$ if $I_P \subseteq I_{P'}$ and $\omega_{P'} \subseteq \omega_P$.

If $P < P'$ we have two possibilities:

A tree $T$ is a set of tiles $P$ such that there exists a tile $P_T$ with $P < P_T$ for all $P \in T$. We do not assume $P_T \in T$.

A 2-tree satisfies $\omega_{2P_T} \subseteq \omega_{2P}$ for all $P \in T$: 
Mass

With $E$ the given set of measure 1 define

$$E_P = E \cap \{x : N(x) \in \omega_P\}$$

For $P$ a set of tiles define

$$\text{mass}(P) := \max_{P \in \mathcal{P}, P' \in \mathcal{P} : P < P'} \frac{|I_{P'} \cap E_P|}{|I_{P'}|}$$

Mass- Proposition

Let $P \subset \overline{P}$. Then $\text{mass}(P) \leq C$. Moreover, $P$ is the union of two sets $P_1$ and $P_2$ with

$$\text{mass}(P_1) \leq \text{mass}(P)/2$$

and $P_2$ is the union of trees $T \in \mathcal{T}$ such that

$$\sum_{T \in \mathcal{T}} |I_{P_T}| \leq C \text{mass}(P)^{-1}.$$
Proof of Mass Proposition

Let $P_2$ be the set of $P \in P$ with

$$\text{mass}\{\{P\}\} \geq \frac{\text{mass}(P)}{2}.$$

Then $P_1 = P \setminus P_2$ is as required.

Let $P'$ be the set of maximal tiles in $\overline{P}$ s.t.:

$$|I_{P'} \cap E_{P'}|/|I_{P'}| \geq \frac{\text{mass}(P)}{2}.$$

Then $P_2$ is a union of trees $T$ with $P_T \in P'$.

The tiles $P' \in P'$ are pairwise disjoint.
Hence the sets $I_{P'} \cap E_{P'}$, are pairwise disjoint.

$$\sum_{P' \in P'} |I_P| \leq \sum_{P' \in P'} \frac{2|I_{P'} \cap E_{P'}|}{\text{mass}(P)} \leq \frac{2}{\text{mass}(P)}$$
A stronger mass proposition

Define the weight function

\[ w(x) = (1 + |x|)^{-100} \]

For a tile \( P \) define

\[ w_P = T_c(I_P) D_{|I_P|}^1 w \]

Then the mass proposition continues to hold for the modified definition of mass as

\[ \max_{P \in \mathcal{P}, P' \in \mathcal{P}: P < P'} \int_{E_{P'}} w_{P'}(x) \, dx . \]

Proof: Apply the old result to the rectangles

(This page is superfluous in the Walsh model)
Energy

Recall \( f \in \mathcal{S}(\mathbb{R}) \) with \( \|f\|_2 = 1 \). Define

\[
\text{energy}(\mathbf{P}) := \max_T \left( |I_{P_T}|^{-1} \sum_{P \in T} |\langle f, \phi_1 P \rangle|^2 \right)^{\frac{1}{2}}
\]

where the max is taken over all 2-trees \( T \subset \mathbf{P} \).

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Energy-Proposition

Let \( \mathbf{P} \subset \mathbf{P} \). Then \( \mathbf{P} \) is the union of two sets \( \mathbf{P}_1 \) and \( \mathbf{P}_2 \) with

\[
\text{energy}(\mathbf{P}_1) \leq \text{energy}(\mathbf{P})/2
\]

and \( \mathbf{P}_2 \) is the union of trees \( T \in \mathbf{T} \) such that

\[
\sum_{T \in \mathbf{T}} |I_{P_T}| \leq C \text{energy}(\mathbf{P})^{-2}.
\]
Idea of Energy Proposition

Remove a 2- tree $T \in \mathcal{P}$ such that

$$\left(\left|I_{P_T}\right|^{-1} \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}} \geq \text{energy}(\mathcal{P})/2,$$

the center of $\omega_{P_T}$ is minimal (primary goal), and the tree is maximal w.r.t. set inclusion (secondary goal). Iterate until the remainder set $\mathcal{P}_1$ has half the energy of $\mathcal{P}$.

The rectangles $I_P \times \omega_{1P}$ of all $P \in \mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$ are pairwise disjoint. For assume not, then

If the corresponding $\phi_{1P}$ were orthogonal:

$$\sum_T |I_{P_T}| \leq 4 \text{energy}(\mathcal{P})^{-2} \sum_T \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \leq 4\text{energy}(\mathcal{P})^{-2}$$
End of Proof

Decompose $\mathbf{P}$ into sets $\mathbf{P}_n$ with $n \in \mathbb{Z}$ s.t.:

\[
\text{mass}(\mathbf{P}_n) \leq \min(C, 2^{2n}) ,
\]

\[
\text{energy}(\mathbf{P}_n) \leq 2^n ,
\]

and $\mathbf{P}_n$ is a union of trees $T \in \mathbf{T}_n$ such that

\[
\sum_{T \in \mathbf{T}_n} |I_{P_T}| \leq 2^{-2n} .
\]

We will see later that for each tree $T$:

\[
\sum_{P \in T} \left| \langle f, \phi_1P \rangle \langle \phi_1P, 1_{E_{2P}} \rangle \right| 
\leq C \text{energy}(T) \text{mass}(T) \left| I_{P_T} \right| .
\]

Hence we have

\[
\sum_{n \in \mathbb{Z}} \sum_{T \in \mathbf{T}_n} \sum_{P \in T} \left| \langle f, \phi_1P \rangle \langle \phi_1P, 1_{E_{2P}} \rangle \right| 
\leq C \sum_{n \in \mathbb{Z}} \min(2^n, 2^{-n}) \leq C .
\]

This gives Carleson’s theorem.
Idea of Tree Estimate

Fix a tree $T$. Let $J$ be a maximal dyadic interval such that $2J$ does not contain any $I_P$ with $P \in T$. Then

$$J \cap \bigcup_{P \in T : J \subset I_P} E_P \leq C \text{mass}(T) |J| .$$

We have to estimate for certain $|\epsilon_P| = 1$

$$\left| \int \sum_{P \in T} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P 1_{E_2 P} \right|$$

Pretending $\phi_1 P$ is supported in $I_P$:

$$\sum_J \left\| \sum_{P \in T} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P 1_{E_2 P} \right\|_{L^1(J)} \leq C \text{mass}(T) \sum_J |J| \left\| \sum_{P \in T} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P 1_{E_2 P} \right\|_{L^\infty(J)}$$
The 1-Tree

For $j = 1, 2$ let $T_j$ be the set of $T \in \mathbf{T}$ with such that the center of $\omega_{PT}$ is contained in $\omega_{jP}$.

Consider the tree $T_1$. The sets $I_P \times \omega_{2P}$ are pairwise disjoint, hence so are the sets $E_{2P}$.

We pretend that $\phi_{1P}$ is supported in $I_P$ and hence in $I_{PT}$ for all $P \in T_1$. Since $\{P\}$ is a 2-tree for each $P$, we have

$$
\sum_{J} |J| \left\| \sum_{P \in T_1} \epsilon_P \langle f, \phi_{1P} \rangle \phi_{1P} 1_{E_{2P}} \right\|_{L^\infty(J)}
\leq C \sum_{J \subset I_{PT}} |J| \text{energy}(T)
\leq C \text{energy}(T) |I_{PT}|
$$

This gives the tree estimate for $T_1$. 
The 2-Tree

Assume w.l.o.g. that the center of $\omega_{P_T}$ is 0.

Then $\phi_{1P}$ has mean 0 for each $P \in T_2$. We pretend that $\phi_{1P}$ is the Haar function on $I_P$ if $P \in T_2$.

Fix $J \subset I_T$. For each $x \in J$ there are numbers $|J| < |A_x| < |B_x|$ such that for all $P \in T_2$

$$N(x) \in \omega_{2P} \iff A_x \leq |I_P| \leq B_x,$$

$$x \in E_{2P} \iff A_x \leq |I_P| \leq B_x.$$
A Maximal Truncated Singular Integral

We have pointwise on an interval $J$

$$\left| \sum_{P \in T_2} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P^1 E_{2P} \right|$$

$$\leq 2 \max_{A: |J| < A} \left| \sum_{P \in T_2: |I_P| \geq A} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P \right|$$

$$\leq 2 \max_{I: J \subset I} \frac{1}{|I|} \left| \int_I \sum_{P \in T_2} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P \right|$$

With the Hardy-Littlewood maximal function $M$:

$$\sum_J |J| \left\| \sum_{P \in T_2} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P^1 E_{2P} \right\|_{L^\infty(J)}$$

$$\leq C \left\| M \left( \sum_{P \in T_2} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P \right) \right\|_{L^1(I_{P_T})}$$

$$\leq C |I_{P_T}|^{1/2} \left\| \sum_{P \in T_2} \epsilon_P \langle f, \phi_1 P \rangle \phi_1 P \right\|_2$$

$$\leq C |I_{P_T}| \text{energy}(T)$$.
Proof of Energy Proposition

We have to prove Bessel’s inequality despite leakage of $\phi_{1P}$ in spatial direction. This is done by a variant of Schur’s test.

$$\left( \sum_{P \in P_2} |\langle f, \phi_{1P} \rangle|^2 \right)^2 \leq \left\| \sum_{P} \langle f, \phi_{1P} \rangle \phi_{1P} \right\|_2^2$$

$$\leq \sum_{P,P':\omega_P = \omega_{P'}} \langle f, \phi_{1P} \rangle \langle \phi_{1P}, \phi_{1P'} \rangle \langle \phi_{1P'}, f \rangle$$

$$+ 2 \sum_{P,P':\omega_P \subseteq \omega_{1P'}} \langle f, \phi_{1P} \rangle \langle \phi_{1P}, \phi_{1P'} \rangle \langle \phi_{1P'}, f \rangle$$

$$\leq 2 \sum_{P} |\langle f, \phi_{1P} \rangle|^2 \max_{P'} \sum_{\omega_P = \omega_{P'}} |\langle \phi_{1P}, \phi_{1P'} \rangle|$$

$$+ 2 \sum_{T} \sum_{P \in T} \langle f, \phi_{1P} \rangle \sum_{P':\omega_P \subseteq \omega_{1P'}} \langle \phi_{1P}, \phi_{1P'} \rangle \langle \phi_{1P'}, f \rangle$$
The Off-Diagonal Term

\[ \text{off.diag.term} \leq 2 \sum_{T} \left( \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}} \times \]

\[ \left( \sum_{P \in T} \left| \sum_{P': \omega_p \subset \omega_{1P'}} \langle \phi_{1P}, \phi_{1P'} \rangle \langle \phi_{1P'}, f \rangle \right|^2 \right)^{\frac{1}{2}} \]

But each \( \{P'\} \) is a 2-tree by itself and thus

\[ |\langle \phi_{1P'}, f \rangle| \leq |I_{P'}|^2 \text{energyP} \]

Thus by selection of \( T \):

\[ \text{off.diag.term} \leq 2 \sum_{T} \left( \sum_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}} \times \]

\[ \left( \sum_{P \in T} \left| \sum_{P': \omega_p \subset \omega_{1P'}} |\langle \phi_{1P}, \phi_{1P'} \rangle| I_{P'}^{\frac{1}{2}} \left| I_{PT} \right|^{-\frac{1}{2}} \right|^2 \right)^{\frac{1}{2}} \]