Mathematics 245A
Terence Tao
Final, March 23, 2004

Instructions: Do all seven problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your final score posted. Good luck!

Name: _______________________

Nickname: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points). ____________
Problem 2 (10 points). ____________
Problem 3 (10 points). ____________
Problem 4 (10 points). ____________
Problem 5 (10 points). ____________
Problem 6 (10 points). ____________
Problem 7 (10 points). ____________
Total (70 points): ____________________
**Problem 1.** Let \((X,d)\) be a metric space. We say that a point \(x \in X\) is a *limit point* if it is the limit of some sequence \(x_1, x_2, \ldots\) in \(X\) such that \(x_n \neq x\) for all \(n\). We say that \(X\) is *perfect* if every point is a limit point.

Show that every metric space which is both complete and perfect, must be uncountable. (Hint: use the Baire category theorem).
**Problem 2.** Let $X$ be a Banach space. Show that the weak and weak-* topologies on $X^*$ coincide if and only if $X$ is reflexive. (One direction is easy; the other requires the Hahn-Banach theorem.)
**Problem 3.** Let $H$ be a Hilbert space, and let $(e_\alpha)_{\alpha \in A}$ be an orthonormal basis for $H$ (which may be finite, countable, or uncountable). Let $x_1, x_2, x_3, \ldots$ be a sequence of vectors in $H$ which are bounded (i.e. there exists an $M > 0$ such that $\|x_n\| \leq M$ for all $n$). Let $x$ be another vector in $H$.

Show that the sequence $x_n$ is weakly convergent to $x$ if and only if we have $\lim_{n \to \infty} \langle x_n, e_\alpha \rangle = \langle x, e_\alpha \rangle$ for all $\alpha \in A$. 


Problem 4. Let $W$ be a vector space. Let $A$ be an index set, and for each $\alpha \in A$, let $V_\alpha$ be a subspace of $W$ which is equipped with a norm $|||V_\alpha$. Suppose that for each $\alpha$, the norm $|||V_\alpha$ turns $V_\alpha$ into a Banach space. Define a new vector space $U$ by

$$U := \{ x \in \bigcap_{\alpha \in A} V_\alpha : \sum_{\alpha \in A} ||x||_{V_\alpha} < \infty \}$$

and equip this space with the norm

$$||x||_U := \sum_{\alpha \in A} ||x||_{V_\alpha}.$$ 

Show that $|||U$ is indeed a norm, and this norm turns $U$ into a Banach space.
Problem 5. Let $H$ be a Hilbert space. Recall that if $M$ is a closed subspace of $H$, then we can define a linear operator $P_M : H \to H$ by defining $P_M x$ to be the element of $M$ such that $x - P_M x \in M^\perp$ (see Problem 58 of Chapter 5).

Suppose that we have a sequence $M_1, M_2, \ldots$ of closed subspaces of $H$ such that $M_n \subseteq M_{n+1}$ for all $n$. Let $M_\infty := \bigcup_{n=1}^{\infty} M_n$, i.e. we let $M_\infty$ be the closure of the union of all the subspaces $M_n$. Show that $P_{M_n}$ converges to $P_M$ in the strong operator topology (see page 169 of textbook).
Problem 6. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 < p < \infty$. Let $f : X \to \mathbb{C}$ be a measurable function. Show that $f$ lives in weak $L^p$ (i.e. $\|f\|_p < \infty$) if and only if there exists an $M > 0$ such that for every measurable set $E$ of finite measure, the integral $\int_E |f| \, d\mu$ is absolutely convergent and obeys the estimate

$$\int_E |f| \, d\mu \leq M \mu(E)^{1/p}.$$  

(Hint: You may wish to estimate the distribution function $\lambda_{f|_E}$ of the restriction $f|_E$ of $f$ to $E$ in terms of the distribution function $\lambda_f$ of $f$ itself.)
Problem 7. Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces, and let \(1 < p, q < \infty\). Let 
\[ T : L^p(X) \to L^q(Y) \] 
be a continuous linear operator which has the form 
\[ Tf(y) = \int_X K(x, y) f(x) \, d\mu(x) \] 
for all simple functions \(f\). (The restriction to simple functions is to ensure that the above integral is actually convergent for almost every \(y\), thanks to Fubini’s theorem). Let \(T^* : L^q(Y) \to L^p(X)\) be the adjoint of \(T\) as defined in Exercise 22 of Section 5.2, and where we have identified the dual of \(L^p(X)\) with \(L^{p'}(X)\), and the dual of \(L^q(Y)\) with \(L^{q'}(Y)\) in the usual manner (cf. Theorem 6.15). Show that 
\[ T^* g(x) = \int_Y \overline{K(x, y)} g(y) \, d\nu(y) \] 
for all simple functions \(g\).