Problem 1. (True/False, 1 pt each) Mark your answers by filling in the appropriate box next to each question.

(a) (False) The reduced row echelon form of the matrix
\[
\begin{bmatrix}
2 & 3 & 7 \\
1 & 4 & 8 \\
6 & 5 & 9
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
Note: the given matrix is not even in RREF.

(b) (True) If A is an invertible \( n \times n \) matrix, then the rank of A is \( n \).

(c) (True) There exists a matrix \( A \) for which \( \ker A \) is the same as the image of \( A \).

(d) (False) If \( AB = BA \) for two \( n \times n \) matrices \( A, B \), then \( A \) or \( B \) must be the identity matrix.

(e) (True) Assume that the reduced row echelon form of \( A \) is
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Does the system of linear equations \( A\vec{x} = 0 \) have more than one solution?

(f) (True) If \( A \cdot A \cdot A \) is the identity matrix, then \( A \) is invertible. Note: Actually \( A^{-1} = A^2 \) in this case, since \( A \cdot A^2 = A \cdot A \cdot A = I \).

(g) (False) The columns of a matrix \( A \) always form a basis for the image of \( A \). Note: the columns may be linearly dependent, thus not a basis (but they do form a spanning set for the image of \( A \)).

(h) (False) There exists a system of linear equations having exactly 2 distinct solutions.

(i) (False) Every nonzero \( 4 \times 4 \) matrix has an inverse.

(j) (False) The set \( \{(x, y) : x^2 + y^2 = 1\} \) is a subspace of \( \mathbb{R}^2 \).
Problem 2. (10 pts) Let \( A = \begin{bmatrix} 1 & 1 & 4 & 7 & 1 \\ 1 & 2 & 5 & 8 & 0 \\ -1 & 3 & 6 & 9 & 1 \end{bmatrix} \). (a) Are the columns of \( A \) linearly independent? (b) Find a basis for the image of \( A \). (c) Find a basis for the kernel of \( A \).

Solution. (a) The columns of \( A \) cannot be linearly independent. If they were, these five columns would form a basis for a five-dimensional linear space, and it would follow that the image of \( A \) is five-dimensional. But the image of \( A \) is contained in \( \mathbb{R}^3 \), all of whose subspaces have dimension at most 3.

OR: We can compute
\[
\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}.
\]
From this we see that the last 2 columns are redundant.

(b) Looking at \( \text{rref}(A) \) we see that the first three columns of \( A \) form a basis for the linear span of the image of \( A \). Thus a basis is given by:
\[
\left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right).
\]

(c) Solving \( Ax = 0 \) gives us (using \( \text{rref}(A) \) computed above): we have two free variables, \( x_4 = t_1 \) and \( x_5 = t_2 \) and equations
\[
\begin{cases}
  x_1 = t_2 \\
  x_2 = t_1 + 2t_2 \\
  x_3 = -2t_1 - t_2
\end{cases}
\]
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.
\]
From this we see that a basis is given by the vectors
\[
\left( \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).
\]
Problem 3. (10 pts) Let $P$ be the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$. (a) Find a matrix $A$ so that $P$ is the kernel of $A$. (b) Find a matrix $B$ so that $P$ is the image of $B$. (c) Find a basis for $P$ (you may wish to use your results from either (a) or (b)).

Solution. The equation for $P$ can be written as $[\begin{array}{ccc} 1 & 2 & 3 \end{array}] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. (a) If we let $A = [\begin{array}{ccc} 1 & 2 & 3 \end{array}]$ then we precisely get that $P = \ker A$. (c) Since $\dim P = 2$ (e.g. by the rank-nullity theorem: clearly the image of $A$ is 1-dimensional and so its kernel $P$ is 2-dimensional), a basis would consist of any two linearly independent vectors in $P$. By inspection, the vectors

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

belong to $P$. Moreover, they are linearly independent (since they are not proportional and are nonzero), and so they must be a basis for $P$. (b) We can now use our vectors from part (c):

$$B = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

Notes: There are many choices for $A$ in part (a) and $B$ in part (b). In fact, $A$ need not be $1 \times 3$; for instance, the following $A$ would also work:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$  

Similarly, we could add arbitrary vectors from $P$ as extra columns to $B$ and still have that $P$ is the image of $B$. We could also use another basis for $P$ in constructing $B$. 
Problem 4. (10 pts) Find a $2 \times 2$ matrix $A$ with entries real numbers so that $A^2 = -I$ (here $I$ denotes the identity matrix).

Solution. Thinking of $A$ as a linear transformation, we need to find a linear transformation of the plane whose square is $-I$, i.e

$$A^2 = \text{rotation by } 180^\circ.$$

Clearly, rotation by $90^\circ$ would work. So we could take

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Notes: A lot of students found their $A$ by brute force, taking a general $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and determining which conditions on $a, bc, d$ would give $A^2 = -I$. This is also fine, but is harder to work out.
Problem 5. (10 pts) Let \( A \) be an \( n \times m \) matrix, and let \( B \) be an invertible \( m \times m \) matrix. (a) Show that the kernel of \( A \) is the same as the kernel of \( AB \). (b) Determine the rank of \( AB \) in terms of the rank of \( A \).

Solution. There was a typo in part (a) of the problem: the problem should have said: “show that the dimension of the kernel of \( A \) is the same as the dimension of the kernel of \( AB \).” Everybody was given 5 points for part (a), regardless of what they wrote.

As stated, part (a) is incorrect. Consider for example \( A = \begin{bmatrix} 1 & 0 \\ \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

Then \( \ker A \) consists of all vectors of the form \( \begin{bmatrix} t \\ 0 \end{bmatrix} \), while \( AB = \begin{bmatrix} 0 & 1 \\ \end{bmatrix} \) has as kernel the set of vectors of the form \( \begin{bmatrix} t \\ 0 \end{bmatrix} \). So, in general, \( \ker AB \) is not the same as \( \ker A \). Surprisingly, nobody pointed this out during the exam.

The correct solution to parts (a) and (b) would have been to note that since \( B \) is onto, the image of \( A \) is the same as that of \( AB \). Indeed, if \( z \) is in the image of \( A \), then \( z = Ax \) for some \( x \). But then

\[
z = Ax = ABB^{-1}x = ABy, \quad \text{where } y = B^{-1}x,
\]

so \( y \) is in the image of \( AB \) as well. Conversely, if \( z \) is in the image of \( AB \), then \( z = ABx = Ay \) where \( y = Bx \), so that also \( z \) is in the image of \( A \). Thus \( \text{image} A = \text{image} B \). This also proves that

\[
\text{rank} A = \dim \text{image} A = \dim \text{image} AB = \text{rank} AB
\]

answering part (b).

From this we get, using rank-nullity,

\[
\dim \ker A = n - \dim \text{image} A = n - \dim \text{image} AB = \dim \ker AB.
\]

We gave full credit to those students that deduced (b) from knowing that \( \dim \ker A = \dim \ker AB \) and then using the rank-nullity to deduce that

\[
\text{rank} A = n - \dim \ker A = n - \dim \ker AB = \text{rank} AB.
\]

Notes. A few incorrect solutions ought to be mentioned. Some students wrote statements like “because \( B \) is invertible, (pre)composition with \( B \) does not reduce rank”. While this is correct, it needs to be justified [this is the point of the problem!]

Other students wrote because \( \text{rref} (B) = I \), one can conclude that \( \text{rref} (A) = \text{rref} (AB) \). But this equality is not always true (if it were, we would get that \( \ker A = \ker AB \) which does not always hold).

Other students thought that \( \text{rref} (AB) \) were always \( \text{rref} (A) \cdot \text{rref} (B) \). But this is also not true (take e.g. \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \); then \( AB = 0 \) but \( \text{rref} (A) = \text{rref} (B) = A \) and so \( \text{rref} (A) \text{rref} (B) = A^2 \neq 0 = \text{rref} (AB) \).