§8.4, Problems (1, 3, 5), 7 – 17 odd, 18.

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(1-5) Write out the partial fraction decomposition. Do not determine the numerical values of the coefficients

1. \[ \frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{3x+1} \]

3. \[ \frac{2}{x^2 + 3x - 4} = \frac{2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1} \]

5. \[ \frac{x^4}{x^4 - 1} \]

Trying to solve this directly:

\[ \frac{x^4}{(x^2 + 1)(x^2 - 1)} = \frac{x^4}{(x^2 + 1)(x+1)(x-1)} \]

\[ = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1} + \frac{D}{x - 1} \]

Notice that the system of equations

\[
x^4 = (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x - 1)
\]

\[
= (A + C + D)x^3 + (B - C - D)x^2 + (C + D)x + (-C - D)
\]

has no solution. The left-hand side has higher order than anything on the right-hand side. Partial fractions will not be able to rearrange this expression. To remedy this, we bring down the order of the numerator:

\[
\frac{x^4}{x^4 - 1} = \frac{(x^4 - 1) + 1}{x^4 - 1} = 1 + \frac{1}{x^4 - 1}
\]

Now this is something that we can break into partial fractions

\[ 1 + \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1} + \frac{D}{x - 1} \]
Evaluate the integral

7. \[ \int \frac{x}{x-6} \, dx \]
   \[ = \int \frac{(x-6)+6}{x-6} \, dx = \int (1 + \frac{6}{x-6}) \, dx \]
   \[ = x + 6 \ln |x-6| + C \]

9. \[ \int \frac{x-9}{(x+5)(x-2)} \, dx \]
   \[ = \int \frac{x-9}{(x+5)(x-2)} \, dx = \int \left( \frac{A}{x+5} + \frac{B}{x-2} \right) \, dx \]
   \[ = A \ln |x+5| + B \ln |x-2| + C \]

Determining the coefficients:

\[ x - 9 = A(x - 2) + B(x + 5) \]
\[ = (A + B)x + (-2A + 5B) \]

If we equate coefficients, we have the system

\[ 1 = A + B \]
\[ -9 = -2A + 5B \]

which has the solution \( A = 2, B = -1 \). So our final answer is

\[ 2 \ln |x+5| - \ln |x-2| + C = \ln \left| \frac{(x+5)^2}{x+2} \right| + C. \]

11. \[ \int_2^3 \frac{1}{x-1} \, dx \]
   \[ = \int_2^3 \frac{1}{(x-1)(x+1)} \, dx = \int_2^3 \left( \frac{A}{x+1} + \frac{B}{x-1} \right) \, dx \]
   \[ = [A \ln |x+1| + B \ln |x-1|]_2^3 \]

Determining the coefficients:

\[ 1 = A(x - 1) + B(x + 1) \]
\[ = (A + B)x + (-A + B) \]

If we equate coefficients, we have the system

\[ 0 = A + B \]
\[ 1 = -A + B \]

which has the solution \( A = -1/2, B = 1/2 \). So our final answer is

\[ \frac{1}{2} \left[ \ln \left| \frac{x-1}{x+1} \right| \right]_2^3 = \frac{1}{2} \ln \left( \frac{2+1}{2-1} \cdot \frac{3-1}{3+1} \right) = \frac{1}{2} \ln(3/2). \]
13. \( \int \frac{ax}{x^2-b^2} \, dx \)

Oddly, this simplifies to

\[ \int \frac{a}{x-b} \, dx = a \ln |x-b| + C. \]

15. \( \int_0^1 \frac{2x+3}{(x+1)^2} \, dx \)

Let’s clear up the denominator with a \( u \)-substitution:

\[ u = x + 1 \quad du = dx \]

Then our integral becomes

\[ \int_1^2 \frac{2(u-1) + 3}{u^2} \, du = \int_1^2 \frac{2u+1}{u^2} \, du \]

\[ = \int_1^2 \left( \frac{2}{u} + \frac{1}{u^2} \right) \, du = \left[ 2 \ln |u| - \frac{1}{u} \right]_1^2 \]

\[ = 2 \ln(2) + \frac{1}{2} \]

(Notice that I’ve changed the limits of integration from \((0,1)\) to \((1,2)\))

17. \( \int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)} \, dy \)

\[ = \int_1^2 \left( \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3} \right) \, dy \]

\[ = |A \ln |y| + B \ln |y+2| + C \ln |y-3| |_1^2 \]

Determining the coefficients:

\[ 4y^2 - 7y - 12 = A(y+2)(y-3) + B(y)(y-3) + C(y)(y+2) \]

\[ = (A + B + C)y^2 + (-A - 3B + 2C)y - 6A \]

If we equate coefficients, we have the system

\[ 4 = A + B + C \quad -7 = -A - 3B + 2C \quad -12 = -6A \]

which has the solution \( A = 2, B = 9/5, C = 1/5 \). So our final answer is

\[ \left[ 2 \ln |y| + \frac{9}{5} \ln |y+2| + \frac{1}{5} \ln |y-3| \right]_1^2 \]

\[ = 2 \ln(2/1) + \frac{9}{5} \ln(4/3) + \frac{1}{5} \ln(1/2) = \frac{9}{5} \ln(8/3) \]
18. $\int \frac{x^2 + 2x - 1}{x^2 - x} \, dx$

A little rearrangement turns what would be a system of three equations into a system of two equations (which we have solved already).

\[
= \int \frac{(x + 1)(x - 1) + 2x}{x(x + 1)(x - 1)} \, dx = \int \left( \frac{1}{x} + \frac{2}{(x + 1)(x - 1)} \right) \, dx
\]

\[
= \int \left( \frac{1}{x} + 2 \left( \frac{A}{x + 1} + \frac{B}{x - 1} \right) \right) \, dx = \ln |x| + 2(A \ln |x + 1| + B \ln |x - 1|).
\]

From (11), we know that $A = -1/2$ and $B = 1/2$. Our final solution is then

\[
\ln |x| - \ln |x + 1| + \ln |x - 1| + C = \ln \left| \frac{x^2 - x}{x + 1} \right| + C.
\]