Problem 1. State and prove the Bolzano-Weirstrass theorem.

Note: it is likely that you will be asked to give a proof of one of the major theorems on the midterm. Here’s the list of major theorems: limit points of a given sequence are exactly limits of all possible subsequences; Bolzano-Weirstrass; addition/multiplication/composition/etc. of continuous functions are continuous; intermediate value theorem; continuous functions attain their max/min on a closed interval; continuous functions are uniformly continuous on closed intervals; continuous functions are Riemann integrable; properties of Riemann integrals (addition, scalar multiplication).

Solution: see book.

Problem 2. Let \( \{a_n\} \) be a bounded sequence. Show that this sequence has a limit if and only if it has exactly one limit point.

Solution: If \( a_n \to a \), then \( a \) is the unique limit point. Indeed, by a theorem in the book \( c \) is a limit point of \( \{a_n\} \) iff some subsequence \( a_{n_k} \to c \). But any subsequence of \( a_n \) converges to \( a \): given \( \varepsilon > 0 \), there is an \( N \) so that if \( n > N \), \( |a_n - a| < \varepsilon \). Thus if we choose \( K \) so that \( n_K > N \), then for any \( k > K \), \( |a_{n_k} - a| < \varepsilon \). Thus any limit point of \( \{a_n\} \) must be \( a \), and \( a \) is a limit point, since \( \{a_n\} \) is its own subsequence, and converges to \( a \).

Conversely, suppose that \( a \) is a limit point of \( \{a_n\} \). We claim that \( a_n \to a \).

If not, there is an \( \varepsilon > 0 \) so that for all \( N \), there is an \( n > N \) for which \( |a_n - a| > \varepsilon \).

We now construct a subsequence \( \{a_{n_k}\} \) of \( \{a_n\} \) inductively. Let \( \varepsilon \) be as above, and apply the above assertion with \( N = 1 \). Then there is an \( n_1 > N \) so that \( |a_{n_1} - a| > \varepsilon \). Having constructed \( n_1, \ldots, n_{k-1} \), apply the assertion above with \( N = n_{k-1} \). Then there is an \( n_k > K \) so that \( |a_{n_k} - a| > \varepsilon \). Since \( n_k > K_k = n_{k-1} \), the indices \( n_k \) are monotone increasing and so \( \{a_{n_k}\} \) is a subsequence.

Since \( \{a_n\} \) is bounded, so is \( \{a_{n_k}\} \). By the Bolzano-Weirstrass theorem, there is a limit point, \( b \), of the sequence \( a_{n_k} \). Thus there is a subsequence, of \( \{a_{n_k}\} \), converging to \( b \). Then, since \( |a_{n_k} - a| > \varepsilon \) for all \( k \), the same is true for our subsequence of \( \{a_{n_k}\} \). It follows that \( |b - a| > \varepsilon \).
A subsequence of \( \{a_{n_k}\} \) is a subsequence of \( \{a_n\} \), so there is a subsequence of \( \{a_n\} \) converging to \( b \). Since \( \{a_n\} \) has only one limit point, it must be that \( b = a \). This is a contradiction, since we proved that \( |b - a| > \varepsilon \).

**Problem 3.** (a) Give the book’s definition of continuity of a function at a point \( c \).

(b) Let \( f(x) \) be given by

\[
   f(x) = \begin{cases} 
   0, & \text{if } x \text{ irrational}; \\
   1, & \text{if } x \text{ rational}. 
   \end{cases}
\
\]

Show that \( f(x) \) is discontinuous at every point \( c \) in \( \mathbb{R} \).

(c) [A little harder] Let \( f(x) \) be defined as follows:

\[
   f(x) = \begin{cases} 
   0, & \text{if } x \text{ is irrational}; \\
   \frac{1}{m}, & \text{if } x \text{ is rational and } x = \frac{n}{m}, m > 0, \text{ is an irreducible fraction}. 
   \end{cases}
\
\]

(we define \( f(0) = 1 \)). Show that \( f \) is continuous at every \( c \) which is irrational, but is not continuous at every \( c \) which is rational.

**Solution:**  (a) see book.

(b) Let \( \varepsilon > 0 \). Choose two sequences so that \( a_n \to c \) and \( b_n \to c \), but \( a_n \) consist of rational numbers, while \( b_n \) consists of irrational numbers. Then \( f(a_n) \to 1 \), while \( f(b_n) \to 0 \). So it is not possible that \( f(x_n) \to f(c) \) for any sequence \( x_n \) so that \( x_n \to c \).

(c) Sketch of proof: if \( a \) is rational, then \( f(a) \neq 0 \) by definition. Choose a sequence \( a_n \to a \), so that \( a_n \) are irrational. Then \( f(a_n) \to 0 \), but \( f(a) \neq 0 \). So \( f \) is not continuous at \( a \).

To prove that \( f \) is continuous at an irrational \( c \) you start by proving the following: given \( M > 0 \) and integer there is a \( \delta > 0 \) so that if \( |c - x| < \delta \) and \( x \) is rational, \( x = n/m \), then \( m > M \).

This can be proved from the fact that there are at most \( \frac{M}{2\varepsilon} \) rational numbers with denominator at most \( M \) in any interval of length \( 2\delta \) (in particular, in \( (c - \delta, c + \delta) \)).

Now, given \( \varepsilon > 0 \), choose \( M \) so that \( \varepsilon > \frac{1}{M} \). For that \( M \), choose \( \delta \) as above. Then if \( |c - x| < \delta \), then either \( x \) is rational and \( x = n/m \) with \( m \geq M \), or \( x \) is irrational. In the former case, \( |f(x)| \leq \frac{1}{M} < \varepsilon \); in the latter case, \( |f(x)| = 0 < \varepsilon \). Thus in any case \( |f(x) - f(c)| < \varepsilon \), since \( f(c) = 0 \).

Using a theorem in the book, it follows that \( f \) is continuous at \( c \).

**Problem 4.** Give an example of a function that is continuous on the open interval \((0, 1)\), but is not uniformly continuous on \((0, 1)\). Prove that your example works.
Solution: let $f(x) = \frac{1}{2}$. Let $\varepsilon = 1$. Then for any $\delta > 0$, there are $x, y \in (0, 1)$ so that $|x - y| < \delta$ but $\frac{1}{x} - \frac{1}{y} > 1 = \varepsilon$. Indeed, just take $y = \min(1/2, \delta)$ and $x = y/2$. Then $|x - y| < \delta$, and
\[
\frac{1}{x} - \frac{1}{y} = \frac{2}{y} - \frac{1}{y} = \frac{1}{y} \geq 2 \geq 1.
\]

Problem 5. Let $f(x)$ be a polynomial in $x$ of odd degree. Show that $f(c) = 0$ for some $c \in \mathbb{R}$ (hint: use the intermediate value theorem).

Solution: Let $f(x) = ax^p + q(x)$, where $p$ is odd, and $q(x)$ is a polynomial of order strictly less than $n$. By replacing $f(x)$ with $\frac{1}{n}f(x)$ (which has the same roots as $f(x)$) we may as well assume that $a = 1$. Then
\[
\frac{f(n)}{n^p} = 1 + \frac{q(n)}{n^p} \rightarrow 1
\]
as $n \rightarrow \infty$. If we let $\varepsilon = \frac{1}{2}$, this means that for some $N$ and all $n > N$,
\[
\left|\frac{f(n)}{n^p} - 1\right| < \frac{1}{2}.
\]

But this means that $f(n)/n^p$ is positive for some $n$; thus $f(n)$ is positive for some $n$. Denote this $n$ by $n_+$.

Applying a similar reasoning to the sequence $f(-n)/(-n)^p$, which also converges to 1, we find that for some $n$, $f(-n)/(-n)^p$ is also positive. Since $p$ is odd, this means that $f(-n)$ is negative for some $n$. Denote this $n$ by $n_-$. Consider now $f$ on the interval $I = [-n_-, n_+]$. Since $f$ is continuous on all of $\mathbb{R}$, it is continuous on $I$. We have that $f$ is positive and negative at the two endpoints of the interval. By the intermediate value theorem, it follows that $f$ must be zero somewhere on $I$.

Problem 6. Let $f(x)$ be a function so that $f(x) \geq 0$ on $[a, b]$. (a) Prove that if $f$ is continuous and $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) Give an example of an intergrable function $f$, which is not continuous, $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x) \, dx = 0$, but so that $f(x)$ is not constantly zero on $[a, b]$.

Solution. (a) We'll prove that if $f$ is continuous, $f \geq 0$ and $f \neq 0$, then $\int_a^b f(x) \, dx \neq 0$. This will prove that if $\int_a^b f(x) \, dx = 0$, then $f$ is constantly zero.

We claim that if $f \neq 0$, then there is an interval $[c, d] \subset [a, b], c \neq d$, and a constant $C > 0$, so that $f(x) \geq C$ for all $x \in [c, d]$. Indeed, since
\( f \neq 0 \), there is an \( x_0 \in [a, b] \), so that \( f(x_0) > 0 \). Let \( \varepsilon = \frac{f(x_0)}{2} > 0 \).

We now apply the \( \varepsilon, \delta \)-criterion of continuity of \( f \) with this \( \varepsilon \). Then there is a \( \delta > 0 \) so that \( |f(x) - f(x_0)| \leq \varepsilon \) if \( |x - x_0| \leq \delta \). Let \( c = x_0 - \delta, \ d = x_0 + \delta \) and \( C = f(x_0)/2 \). Then for any \( x \in [c, d] \), we have \( |x - x_0| \leq \delta \), so that \( |f(x) - f(x_0)| \leq \varepsilon = f(x_0)/2 \). But then \( f(x) \geq f(x_0)/2 = C \), as claimed.

Now consider the partition \( P = (c, d) \) of the interval \([a, b]\). We have

\[
L_P(f) = m_1(c - a) + m_2(d - c) + m_3(b - d),
\]

where \( m_1 \) the the infimum of \( f \) on \([a, c]\), \( m_2 \) is the infimum of \( f \) on \([c, d]\) and \( m_3 \) is the infimum of \( f \) on \([d, b]\). But by construction \( m_2 \geq C > 0 \), and \( m_1, m_2 \geq 0 \), since \( f \geq 0 \). Thus

\[
L_P(f) \geq m_2(d - c) \geq C(d - c) > 0.
\]

Thus

\[
\int_a^b f(x)dx \geq L_P(f) > 0.
\]

(b) (Sketch) Let \( f(x) = 0 \) for \( x \in (0, 1] \), and \( f(0) = 1 \). Then \( f \) is not zero, but \( \int_0^1 f(x)dx = 0 \) (prove this yourself!).