Math 110B Midterm 1

Wednesday, April 29, 2015

Name:

Student ID:

Signature:

You must put all your answers in the spaces provided on the page of the problem. You must show a method of solution to obtain credit for a problem.

No calculators authorized
1. Let $\text{GL}_2(\mathbb{Z})$ be the group of invertible 2 by 2 matrices with integer coefficients and let $\text{SL}_2(\mathbb{Z})$ be the set of 2 by 2 matrices with integer coefficients and determinant 1

$$
\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}.
$$

a. Show that $\text{SL}_2(\mathbb{Z})$ is a subgroup of the group $\text{GL}_2(\mathbb{Z})$. 
b. Consider the matrices
\[ a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \]
Show that \( a \) and \( b \) have finite order and determine the order of those matrices.

c. Show that \( ab \) has infinite order.
2. Consider the element $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$ of $S_5$.

a. Write $\sigma$ as a product of cycles with disjoint supports.

b. Determine the elements of the subgroup of $S_5$ generated by $\sigma$. 
3. Determine all the generators of the group $\mathbb{Z}_{12}$. 
4. Show that the groups $\mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_4 \times \mathbb{Z}_3$ are not isomorphic.
5. Let $G$ be a group and let $g \in G$ be an element of finite order $n$. Show that there is a unique homomorphism of groups $f : \mathbb{Z}_n \to G$ such that $f(1) = g$. 
6. Let $G$ be a group, let $H$ and $K$ be subgroups of $G$. Let $HK$ be
the set of elements of $G$ of the form $hk$ with $h \in H$ and $k \in K$.

Show that $HK$ is a subgroup of $G$ if and only if $HK = KH$. 