2. Let $x_n$ be defined by $x_1 = 1$ and $x_{n+1} = 1/(3+x_n)$ for $n = 1, 2, \ldots$
Prove that $(x_n)$ converges and determine its limit.

Hint: Show first that $|x_{n+1} - x_n| \leq (1/9)|x_n - x_{n-1}|$

Proof: First

$$
x_{n+1} - x_n = \frac{1}{3 + x_n} - \frac{1}{3 + x_{n-1}} = \frac{x_{n-1} - x_n}{(3 + x_n)(3 + x_{n-1})} = 
$$

Second, the defining relations $x_1 = 1$, $x_{n+1} = 1/(3+x_n)$, imply that $x_n > 0$ for all $n$. Consequently $(3 + x_n)(3 + x_{n-1}) > 9$, and

$$
|x_{n+1} - x_n| < (1/9)|x_n - x_{n-1}|.
$$

Iterating this inequality we obtain

$$
|x_3 - x_2| < \frac{1}{9}|x_2 - x_1| = \frac{1}{9\cdot4}
$$

$$
|x_4 - x_3| < \frac{1}{9}|x_3 - x_2| < \frac{1}{9^2\cdot4^2}
$$

$$
\ldots
$$

$$
|x_n - x_{n-1}| < \frac{1}{9^{n-2}\cdot4^{n-2}}
$$

This, in turn, implies that that $(x_n)$ is a Cauchy sequence (proof?) So, $(x_n)$ converges

To find what it converges to take limits:

$$
lx_{n+1} = \lim_{n \to \infty} \frac{1}{3 + x_n} \Rightarrow x = \frac{1}{3 + x}
$$

$$
x^2 + 3x - 1 = 0 \Rightarrow 
$$

$$
x = \frac{-3 \pm \sqrt{9 + 4}}{2}
$$

Finally,

$$
x > 0 \Rightarrow x = \frac{-3 + \sqrt{13}}{2}.
$$