9. Give a direct proof that \( f(x) = \sin(x) \) is uniformly continuous on \( (-\infty, \infty) \). That is, given an \( \varepsilon > 0 \) find a \( \delta > 0 \) such that

\[
\text{if } |x - y| < \delta \text{ then } |\sin(x) - \sin(y)| < \varepsilon.
\]

Proof: Recall first that \( |\sin(h)| < |h| \) for small \( h \).

Recall next

\[
\begin{align*}
\sin(A + B) &= \sin(A)\cos(B) + \sin(B)\cos(A) \\
\sin(A - B) &= \sin(A)\cos(B) - \sin(B)\cos(A) \\
\Rightarrow \quad \sin(A + B) - \sin(A - B) &= 2\sin(B)\cos(A)
\end{align*}
\]

Set

\[
\begin{align*}
A + B &= x \\
A - B &= y
\end{align*}
\]

Then \( 2A = x + y \) and \( 2B = x - y \), and

\[
\sin(A + B) - \sin(A - B) = \sin(x) - \sin(y) = 2\sin((x - y)/2)\cos((x + y)/2)
\]

That is,

\[
|\sin(x) - \sin(y)| = |2\sin((x - y)/2)\cos((x + y)/2)| \leq |x - y|.
\]

So, given \( \varepsilon > 0 \), set \( \delta = \varepsilon \). Then, for any \( x \) and \( y \),

\[
|x - y| < \delta = \varepsilon \Rightarrow |\sin(x) - \sin(y)| < \varepsilon,
\]

So, \( \sin(x) \) is uniformly continuous on \( (-\infty, \infty) \).