Review, Problem 8

**Prove that**

\[
1 + \cos(x) + \cos(2x) + \ldots + \cos((n-1)x) = \\
\cos((n-1)x/2) \frac{\sin(nx/2)}{\sin(x/2)}
\]

**Proof:** When \( n = 1 \) the equation reduces to \( 1 = 1 \), which is true.

So, going on to the inductive step we get

\[
1 + \cos(x) + \cos(2x) + \ldots + \cos((n-1)x) + \cos(nx) = \\
\cos((n-1)x/2) \frac{\sin(nx/2)}{\sin(x/2)} + \cos(nx)
\]

Thus the problem reduces to justifying that

\[
\cos((n-1)x/2) \frac{\sin(nx/2)}{\sin(x/2)} + \cos(nx) = \cos(nx/2) \frac{\sin((n+1)x/2)}{\sin(x/2)}.
\]

Multiplying through by \( \sin(x/2) \) we get the equivalent equation

\[
(1) \quad \sin(nx/2) \cos((n-1)x/2) + \sin(x/2) \cos(nx) = \sin((n+1)x/2) \cos(nx/2).
\]

To finish apply the trigonometric identity

\[
\sin(A) \cos(B) = (\sin(A + B) + \sin(A - B))/2.
\]

We get, for the left side of equation (1):

\[
\sin(nx/2) \cos((n-1)x/2) + \sin(x/2) \cos(nx) = \\
\frac{\sin(nx/2 + (n-1)x/2) + \sin(nx/2 - (n-1)x/2)}{2} + \frac{\sin(x/2 + nx) + \sin(x/2 - nx)}{2} = \\
\frac{\sin((n-1/2)x + \sin(x/2)}{2} + \frac{\sin((n+1/2)x) - \sin((n-1/2)x)}{2} = \\
\frac{\sin(x/2) + \sin((n+1/2)x)}{2}
\]
Then, going to the right side of equation (1)
\[
\sin((n+1)x/2)\cos(nx/2) = \\
\frac{\sin((n+1)x/2+nx/2) + \sin((n+1)x/2-nx/2)}{2} = \\
\frac{\sin((n+1/2)x) + \sin(x/2)}{2}
\]

So, equation (1) is valid, which justifies the inductive step.