Review, Problem 7

Prove that

\[(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)\]

where, as usual, \(f^{(k)}(x)\) is the \(k^{th}\) derivative of \(f(x)\).

Proof: When \(n = 1\) the formula is just the product rule for derivatives:

\[(f(x)g(x))^{(1)} = f^{(1)}(x)g(x) + f(x)g^{(1)}(x)\]

So suppose the result is true for \(n\). Then, going on to the inductive step:

\[(f(x)g(x))^{(n+1)} = \frac{d((f(x)g(x))^{(n)})}{dx} = \sum_{k=0}^{n} \binom{n}{k} \frac{df^{(n-k)}(x)g^{(k)}(x)}{dx} = \]

\[
\sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x)g^{(k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k+1)}(x) = 
\]

Now, the very last sum can be re-indexed as

\[
\sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x)g^{(k)}(x) = \sum_{k=0}^{n} \binom{n}{k+1} f^{(n+1-(k+1))}(x)g^{(k+1)}(x) = 
\]

\[
\sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)}(x)g^{(k)}(x)
\]
So, assembling, we have

\[(f(x)g(x))^{(n+1)} = \]

\[\sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(x)g^{(k)}(x) + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)}(x)g^{(k)}(x) = \]

\[f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \left[ \binom{n}{k} + \binom{n}{k-1} \right] f^{(n+1-k)}(x)g^{(k)}(x) + f(x)g^{(n+1)}(x)\]

\[\sum_{k=1}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x)g^{(k)}(x) \]

which completes the proof of the inductive step