Show that \( f(x) = \frac{1}{x} \) is not uniformly continuous on \((0, \infty)\), but is uniformly continuous on any interval \([\mu, \infty)\) where \( \mu > 0 \).

Proof: Suppose that \( f \) is uniformly continuous on \((0, \infty)\). Then, given any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
\text{if } x > 0, y > 0, \quad |x - y| < \delta \quad \text{then} \quad |f(x) - f(y)| < \varepsilon.
\]

So, set \( x_n = \frac{1}{n} \) and \( y_n = \frac{1}{2n} \), where \( n \) is to be determined.

Then

\[
|x_n - y_n| = \frac{1}{2n} < \delta \text{ for sufficiently large } n
\]

\[
|f(x_n) - f(y_n)| = \frac{|x_n - y_n|}{x_n y_n} = n > \varepsilon \text{ for large } n
\]

So, (1) can't hold.

The function is uniformly continuous on \([\mu, \infty)\):

\[
|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{x - y}{xy} \right| \leq \frac{|x - y|}{\mu^2}
\]

Take \( \delta = \mu^2 \varepsilon \). Then

\[
|x - y| < \delta = \mu^2 \varepsilon \Rightarrow \\
|f(x) - f(y)| = \left| \frac{x - y}{\mu^2} \right| < \varepsilon
\]