Prove that all cubic polynomials have at least one real root.

Proof: By definition

\[ f(x) = ax^3 + bx^2 + cx + d, \text{ where } a \neq 0. \]

Factoring out a \( x^3 \), we have

\[
 f(x) = ax^3 \left( 1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{x^3} \right) \approx ax^3
\]

with the approximation \( f(x) \approx ax^3 \) holding for large \( |x| \). Thus there are values of \( x \) for which \( f(x) > 0 \) and other values for which \( f(x) < 0 \). Consequently, by the intermediate value theorem, there is at least one \( x_0 \) such that \( f(x_0) = 0 \).