Let $a_1 = \sqrt{2}$ and
\[
a_{n+1} = \sqrt{2 + \sqrt{a_n}}.
\]

(a) Prove that $\sqrt{2} \leq a_n \leq 2$ for all $n$.
(b) Prove that $\{a_n\}$ is a Cauchy sequence.

Proof: (a) By the induction hypothesis
\[
a_{n+1} = \sqrt{2 + \sqrt{a_n}} \geq \sqrt{2}.
\]
\[
a_{n+1} = \sqrt{2 + \sqrt{a_n}} \leq \sqrt{2 + \sqrt{2}} \leq \sqrt{3.5} \leq 2.
\]

(b) Some algebraic manipulation is required: First
\[
a_{n+1}^2 - a_n^2 = (2 + \sqrt{a_n}) - (2 + \sqrt{a_{n-1}})
\]
\[
= (\sqrt{a_n} - \sqrt{a_{n-1}})(\sqrt{a_n} + \sqrt{a_{n-1}})
\]
\[
= \frac{a_n - a_{n-1}}{\sqrt{a_n} + \sqrt{a_{n-1}}}
\]
So
\[
a_{n+1} - a_n = \frac{a_n - a_{n-1}}{(\sqrt{a_n} + \sqrt{a_{n-1}})(a_{n+1} + a_n)}
\]
By part (a)
\[
(\sqrt{a_n} + \sqrt{a_{n-1}})(a_{n+1} + a_n) \geq 2\sqrt{2}\sqrt{2} \geq 4.
\]
Consequently
\[
|a_{n+1} - a_n| = \frac{|a_n - a_{n-1}|}{(\sqrt{a_n} + \sqrt{a_{n-1}})(a_{n+1} + a_n)} \leq \frac{|a_n - a_{n-1}|}{4}
\]
Iterating this inequality:
\[ |a_3 - a_2| \leq \frac{|a_2 - a_1|}{4} \]
\[ |a_4 - a_3| \leq \frac{|a_3 - a_2|}{4} \leq \frac{|a_2 - a_1|}{4^2} \]
\[ \vdots \]
\[ |a_n - a_{n-1}| \leq \frac{|a_2 - a_1|}{4^{n-2}} \]

Then, as in problem 7,
\[ |a_{n+1} - a_{n-1}| = |a_{n+1} - a_n + a_n - a_{n-1}| \]
\[ \leq |a_{n+1} - a_n| + |a_n - a_{n-1}| \]
\[ \leq \frac{1}{4^{n-1}} + \frac{1}{4^{n-2}} \]

and, more generally,
\[ |a_{n+k} - a_{n-1}| \leq \frac{1}{4^{n+k-2}} + \frac{1}{4^{n+k-3}} + \frac{1}{4^{n-2}} \]
\[ < \frac{1}{4^{n-2}} (1 + \frac{1}{4} + \frac{1}{4^2} + \ldots ) \]
\[ = \frac{1}{4^{n-2}} \cdot \frac{4}{3} \]

That is, replacing \( n+k \) by \( m \) and \( n-1 \) by \( n \), for \( m > n \)
\[ |a_m - a_n| < \frac{3}{4^{n-2}} \]

Since \( n \) is arbitrary, it follows that \( \{a_n\} \) is a Cauchy sequence.