Page 33, Problem 2

Supposing that \( a_n \to L \) find \( L \) and given any \( \epsilon > 0 \) find an integer \( N(\epsilon) \) such that \( |a_n - L| < \epsilon \) for all \( n \geq N(\epsilon) \) where

2a.

\[
a_n = 1 + \frac{10}{\sqrt{n}}
\]

Solution: By inspection, \( L = 1 \). Next

\[
|a_n - 1| = \frac{10}{\sqrt{n}} < \epsilon
\]

\[\iff\]

\[
\frac{10}{\sqrt{n}} < \epsilon
\]

\[\iff\]

\[
n > \frac{100}{\epsilon^2}
\]

so take \( N(\epsilon) = 1 + \lfloor 100/\epsilon^2 \rfloor \).

2b.

\[
a_n = 1 + \frac{1}{n^{1/3}}
\]

Again, by inspection, \( L = 1 \). Next

\[
|a_n - 1| < \epsilon
\]

\[\iff\]

\[
\frac{1}{n^{1/3}} < \epsilon
\]

\[\iff\]

\[
n > \frac{1}{\epsilon^3}
\]

so take \( N(\epsilon) = 1 + \lfloor 1/\epsilon^3 \rfloor \).
2c.

\[ a_n = 3 + \frac{1}{2^n} \]

By inspection, \( L = 3 \). Next

\[ |a_n - 3| = \frac{1}{2^n} < \varepsilon \]

\[ \iff \]

\[ \frac{1}{2^n} < \varepsilon \]

\[ \iff \]

\[ 2^n > 1/\varepsilon \]

\[ \iff \]

\[ n \ln(2) > \ln(1/\varepsilon) \]

So, take \( N(\varepsilon) = \ln(1/\varepsilon)/\ln(2) \)

2d.

\[ a_n = \sqrt[n+1]{n} \]

Solution: First, \( a_n > 0 \). Second

\[ |a_n - 1| = \sqrt[n+1]{n} - 1 \]

\[ \iff \]

\[ |a_n - 1| = \left(1 - \sqrt[n+1]{n}\right) \left(1 + \sqrt[n+1]{n}\right) = 1 - \frac{n}{n+1} \]

\[ \iff \]

\[ |a_n - 1| < \frac{1}{n+1} < \frac{1}{n} \]

\[ \iff \]

\[ |a_n - 1| < \frac{1}{n} < \varepsilon \]

But \( 1/n < \varepsilon \) if and only if \( n > 1/\varepsilon \) so take \( N(\varepsilon) = 1 + \lceil 1/\varepsilon \rceil \).