Suppose that $0 \leq f(x) \leq x^2$ for all real $x$.

(a) Prove that $f$ is differentiable at $x = 0$ and $f'(0) = 0$.

(b) Give an example of a function that satisfies this condition and is not continuous for $x \neq 0$. Is the function continuous at $x = 0$.

Solution: (a) First, since $0 \leq f(0) \leq 0^2$ it follows that $f(0) = 0$. Next, since

$$|f(x)/x| \leq x^2/|x| = |x| \to 0 \text{ as } x \to 0$$

we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = 0$$