Problem 1

Show that, for any two \( n \times n \) matrices \( A, B \), if \( A \sim B \), then \( A^3 \sim B^3 \).

\[ \text{Answer.} \] If \( A \sim B \) then there exists some invertible \( S \) such that \( A = S B S^{-1} \). Then \( A^3 = (S B S^{-1})^3 = S B S^{-1} S B S^{-1} S B S^{-1} = S B I_n B S^{-1} = S B^3 S^{-1} \), meaning that \( A^3 \sim B^3 \).

Problem 2

Is the matrix \( A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \) similar to any diagonal matrix?

\[ \text{Answer.} \] The eigenvalues of \( A \) are the roots of \( \det(A - \lambda I_2) = (5 - \lambda^2) - 2 = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) \). So we have two eigenvalues: 4 and 6, with \( \text{almu}(4) = \text{almu}(6) = 1 \). Therefore, we must have \( \text{gemu}(4) = \text{gemu}(6) = 1 \) as well, which is enough to know that there is an eigenbasis. The eigenbasis is also easy to find as \( \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) \). Therefore, \( A \) is diagonalizable, and

\[ A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}. \]