Problem 1

Let \( \hat{u}_1, \ldots, \hat{u}_n \) be an orthonormal basis for \( \mathbb{R}^n \). Show that if \( \vec{x} = a_1 \hat{u}_1 + \ldots + a_n \hat{u}_n \) is the unique way to write \( \vec{x} \) as a linear combination of this basis, then \( a_i = \vec{x} \cdot \hat{u}_i \) for all \( 1 \leq i \leq n \).

Answer. From the definition of the orthonormal basis we have that \( \hat{u}_i \cdot \hat{u}_j \) is 0 if \( i \neq j \) and 1 if \( i = j \). So,
\[
\vec{x} \cdot \hat{u}_i = (a_1 \hat{u}_1 + \ldots + a_n \hat{u}_n) \cdot \hat{u}_i = \sum_{j=1}^{n} a_j (\hat{u}_j \cdot \hat{u}_i) = a_i.
\]

Problem 2

Find the orthogonal projection of the vector \( \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \) onto the space \( V \) with basis \( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

Answer. We find orthonormal basis \( \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \) of \( V \) (we only had to fix the magnitude of the first vector: they were clearly already orthogonal), and use these to do the dot product method or as our columns in a matrix \( Q \), then we can find the projection matrix and projection as
\[
\text{proj}_V \vec{x} = AQQ^T \vec{x} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 1/3 \end{bmatrix} \vec{x} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -3 \end{bmatrix}.
\]